CHAPTER 18

CONTROL SYSTEM DESIGN USING STATE-SPACE METHODS

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1 INTRODUCTION

The advantages of feedback control in achieving desired input/output relationships are well known. Control system theory based on a frequency-domain approach[1] illustrates clearly that the following aspects of single-input–single-output (SISO) system performance can be improved by feedback: (1) the ability to follow reference inputs accurately in the steady state or under transient conditions and (2) the ability to reject disturbance inputs and reduce sensitivity of the overall controlled system behavior to plant parameter variations and modeling errors. For multiple-input–multiple-output (MIMO) systems, the coupling between individual inputs and outputs can be modified in a desired manner, in addition to the performance features already mentioned, by appropriate control system design.[2]

State-space methods for control system design result in solutions that utilize the state of the system most effectively for feedback. The resulting state-variable feedback control systems improve the same aspects of system performance as previously mentioned. However, the available state-space design procedures accommodate some performance specifications more readily than others. For instance, performance specifications in the form of desired

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closed-loop pole locations are readily accommodated. Similarly, performance specifications in the form of an index of performance to be optimized can be accommodated by optimal-control theory if the index of performance belongs to a restricted class of performance measures. In fact, recent efforts in control system design using state-space methods have been directed at enhancing the problem formulation to accommodate a greater variety of performance specifications. In spite of these enhancements, performance specifications such as sensitivity of the controlled system performance to plant parameter variations and modeling errors are accommodated more readily by frequency-domain-based design procedures than by state-space or time-domain-based design procedures. Thus, control system design techniques based on frequency-domain and time-domain approaches should be viewed as being complementary to each other in some ways.

2 THE POLE PLACEMENT DESIGN METHOD

2.1 Regulation Problem

It can be shown that, if a linear time-invariant (LTI) system is completely state controllable and if linear instantaneous state-variable feedback is used, the associated feedback gains can be chosen to place the closed-loop poles of the controlled system at any arbitrarily specified locations in the s- or z-plane,[3] depending on whether the system is continuous time or discrete time. Thus, if the continuous-time and discrete-time systems described by Eqs. (6) and (12), respectively, are completely state controllable and the control law is given by (Figs. 1 and 2)

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{1}$$

then the eigenvalues of the matrices $\mathbf{A} - \mathbf{BK}$ and $\mathbf{F} - \mathbf{GK}$ are the closed-loop pole locations and can be assigned any specified locations in the complex plane by appropriate selection of the gain matrix $\mathbf{K}$. If $\mathbf{K}$ is constrained to be a real matrix, the desired eigenvalues should be specified either as real or as complex-conjugate pairs. The resulting design procedure is referred to as the pole placement method and is useful for regulation problems where the objective of the controller is to return the system to equilibrium conditions following an initial disturbance. Specification of the closed-loop poles is equivalent to specification of the damping and speed of response of the closed-loop system transients as the system returns to equilibrium.

For single-input systems, specification of the desired closed-loop pole locations uniquely specifies the gain vector $\mathbf{K}$. A formula for the gain vector $\mathbf{K}$, convenient to evaluate and applicable to both continuous-time and discrete-time systems, is

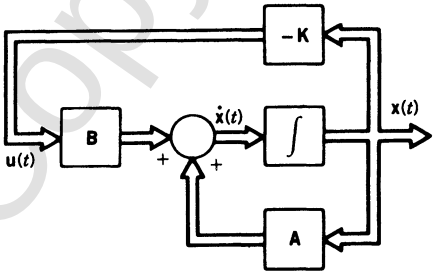


Figure 1 Linear state-variable feedback for continuous-time system.

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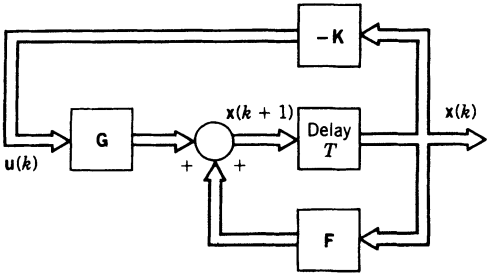


Figure 2 Linear state-variable feedback for discrete-time system.

$$\mathbf{K} = (0 \cdots 0\ 1)(\mathbf{B}\ \mathbf{AB} \cdots \mathbf{A}^{n-1}\mathbf{B})^{-1}\alpha_c(\mathrm{A}) \tag{2}$$

for continuous-time systems and

$$\mathbf{K} = (0 \cdots 0\ 1)(\mathbf{G}\ \mathbf{FG} \cdots \mathbf{F}^{n-1}\mathbf{G})^{-1}\alpha_c(\mathbf{F}) \tag{3}$$

for discrete-time systems. In these equations,

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i \tag{4}$$

for continuous-time systems and $\alpha_c(\mathbf{F})$ is a similar function of $\mathbf{F}$ for discrete-time systems. The α_i's are the coefficients of the desired characteristic equations of the closed-loop systems. For continuous-time systems we have

$$\begin{aligned}\alpha_c(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) \\ &= s^n + \sum_{i=0}^{n-1} \alpha_i s^i = 0\end{aligned} \tag{5}$$

A similar equation describes the discrete-time system characteristic equation. Computer-aided control system design (CACSD) packages supporting state-space methods usually support pole placement designs[4–6] and require only that the designer input information about the system matrices and the desired closed-loop pole locations. The gain vector $\mathbf{K}$ is then computed and output to the designer.

For multi-input systems. specification of the closed-loop poles does not specify the gain matrix $\mathbf{K}$ uniquely. The additional freedom in the gain matrix selection can be used to assign eigenvectors (or generalized eigenvectors) or individual transfer function zeros to improve the transient response to nonzero reference inputs.[7] Alternative criteria for gain matrix selection are optimization of feedback gain magnitudes and stability of the closed-loop system in the absence or failure of some of the inputs. Brogan[8] has outlined a procedure for gain matrix selection for multi-input systems, based on closed-loop eigenvector specification in addition to eigenvalue specification. For continuous-time systems described by Eq. (6) in Chapter 17 and Eq. (20) in this chapter, the feedback gain matrix is given by

$$\mathbf{K} = -(\mathbf{e}_{j_1}\ \mathbf{e}_{j_2} \cdots \mathbf{e}_{j_n})[\psi_{j_1}(s_1)\ \psi_{j_2}(s_2) \cdots \psi_{j_n}(s_n)]^{-1} \tag{6}$$

where the desired closed-loop eigenvalues and the corresponding eigenvectors are s_i and $\psi_{j_i}(s_i)$, $i = 1, \ldots, n$, respectively. The eigenvectors are chosen to be n linearly independent columns from the $n \times nr$ matrix $[\psi(s_1)\ \psi(s_2) \cdots \psi(s_n)]$ where

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$$\psi(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} \tag{7}$$

If the desired s_i are distinct, it will always be possible to find n linearly independent columns as already described. Here $\mathbf{e}_{j_i}$ is defined as the j_ith column of the $r \times r$ identity matrix $\mathbf{I}_r$ and is uniquely determined once j_i is determined. When repeated eigenvalues are desired, the procedure for specifying n linearly independent generalized eigenvectors is different and has been described by Brogan.[8] The results given here are readily applicable to multi-input discrete-time systems described by Eq. (12) in Chapter 17 and Eq. (1) in this chapter. The specified eigenvalues are z_i instead of s_i and the **A** and **B** matrices are replaced by **F** and **G**, respectively. Also, s is replaced by z in Eq. (7). Alternative methods for gain matrix selection for multi-input systems have been described by Kailath.[7]

If a single-input, linear time-variant (LTV) system is completely state controllable, linear state-variable feedback can be used to ensure that the closed-loop transition matrix corresponds to that of atty desired nth-order linear differential equation with time-varying coefficients, The state-variable feedback gains are time varying in general and can be computed using a procedure described by Wiberg.[9]

If the complete state is not available for feedback, linear instantaneous feedback of the measured output can be used to place some of the closed-loop poles at specified locations in the complex plane. If the continuous-time and discrete-time systems described in Chapter 17 by Eqs. (6), (7), (12), and (13), respectively, satisfy the output controllability conditions listed in Table 8 in Chapter 17, then p of the n eigenvalues of the closed-loop system can, approach arbitrarily specified values to within any degree of accuracy but not always exactly. The control law is

$$\mathbf{u} = -\mathbf{K}\mathbf{y} \tag{8}$$

where **K** is a $r \times p$ gain matrix. Brogan[8] has described an algorithm for computing **K**, given the desired values of p closed-loop eigenvalues. The corresponding characteristic equation is

$$\det[s\mathbf{I}_n - \mathbf{A} + \mathbf{B}\mathbf{K}(\mathbf{I}_p + \mathbf{D}\mathbf{K})^{-1}\mathbf{C}] = 0 \tag{9}$$

for continuous-time systems and

$$\det[z\mathbf{I}_n - \mathbf{F} + \mathbf{G}\mathbf{K}(\mathbf{I}_p + \mathbf{D}\mathbf{K})^{-1}\mathbf{C}] = 0 \tag{10}$$

for discrete-time systems.

An alternative approach to control system design in the case of incomplete state measurement is to use an observer or a Kalman filter for state estimation, The estimated state is then used for feedback. This procedure is discussed in Section 5.

The advantages of the pole placement design method already described are that the controller achieves desired closed-loop pole locations without using pole–zero cancellation and without increasing the order of the system. The desired pole locations can be chosen to ensure a desired degree of stability or damping and speed of response of the closed-loop system. However, there is no convenient way to ensure a priori that the closed-loop system satisfies other important performance specifications such as a desired level of insensitivity to plant parameter variations, acceptable disturbance rejection, and compatibility of control effort with actuator limitations. In addition, for single-input LTI systems, instantaneous state feedback of the form given by Eq. (1) does not affect the locations of zeros of the transfer functions between the system input and system outputs.[3] Thus, the pole placement design method does not afford complete control over the system response to the reference input or disturbance inputs. For multi-input systems, the available freedom in the gain matrix selection can be used to assign individual transfer function zeros, in addition to achieving desired closed-loop pole locations. However, systematic procedures to do this are not available. The

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consequence of these limitations of the pole placement method is that the design process involves considerable trial and error.

2.2 Modification for Constant Reference and Disturbance Inputs

The pole placement method described is appropriate for regulation problems. For the case of nonzero reference inputs that may be constant or varying with time, the system outputs are required to follow the reference inputs. The control law, Eq. (1), needs to be modified for such problems. If the output vector $\mathbf{y}$ and the input vector $\mathbf{u}$ have the same dimension and if the $\mathbf{D}$ matrix is zero in Eqs. (7) and (13) in Chapter 17, the modified control law has the form

$$\mathbf{u} = -\mathbf{K}\mathbf{x} + \mathbf{N}\mathbf{y}_d \tag{11}$$

where $\mathbf{y}_d$ is a vector of reference inputs. For constant reference inputs $\mathbf{y}_d$, the error $\mathbf{y}_d - \mathbf{y}$ can be reduced to zero under steady-state conditions by selecting

$$\mathbf{N} = [\mathbf{C}(-\mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B}]^{-1} \tag{12}$$

for continuous-time systems and

$$\mathbf{N} = [\mathbf{C}(\mathbf{I}_n - \mathbf{F} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}]^{-1} \tag{13}$$

for discrete-time systems. The matrices to be inverted on the right-hand sides of the preceding equations exist if and only if the corresponding open-loop transfer matrices $[\mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}]$ and $[\mathbf{C}(z\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}]$ have no zeros at the origin and at $z = 1$, respectively.[3] The $\mathbf{K}$ matrix in Eq. (11) is chosen to give the desired closed-loop poles as before. It should be noted that the gain matrix $\mathbf{N}$ is outside the feedback loop. Hence, the controlled system performance, particularly the steady-state error, would be sensitive to modeling error or error in the elements of the system matrices.

It is well known from classical control theory that integral controller action on the error has the effect of reducing the steady-state error to reference and disturbance inputs. In particular, the steady-state error is reduced to zero for constant reference and disturbance inputs. A similar result can be obtained within the framework of state-variable feedback[3] and will be described for the case where the $\mathbf{y}$ and $\mathbf{u}$ vectors have the same dimension and the $\mathbf{D}$ matrix is zero in Eqs. (7) and (13) in Chapter 17.

Consider the case of constant but unknown disturbance inputs:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \tag{14}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{15}$$

for continuous-time systems and

$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) + \mathbf{G}\mathbf{u}(k) + \mathbf{w}(k) \tag{16}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \tag{17}$$

for discrete-time systems. The state-space equations are augmented by

$$\dot{\mathbf{q}}_e(t) = \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{18}$$

for continuous-time systems and

$$\begin{aligned} \mathbf{q}_e(k+1) &= \mathbf{q}_e(k) + \mathbf{y}(k) \\ &= \mathbf{q}_e(k) + \mathbf{C}\mathbf{x}(k) \end{aligned} \tag{19}$$

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for discrete-time systems. The control law is modified to include feedback of the additional states (Figs. 3 and 4):

$$\mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{K}_q\mathbf{q}_e \tag{20}$$

If the feedback gains $\mathbf{K}$, $\mathbf{K}_q$ are chosen to ensure asymptotic stability of the resulting closed-loop systems, then

$$\lim_{t\to\infty} \dot{\mathbf{q}}_e(t) = 0 \tag{21}$$

for continuous-time systems and

$$\lim_{k\to\infty} \mathbf{q}_e(k) = \text{const} \tag{22}$$

for discrete-time systems, regardless of the value of the disturbance input. When combined with Eqs. (19) and (20), the preceding equations indicate that the output $\mathbf{y}$ and hence the error go to zero in the steady state for continuous-time and discrete-time systems, respectively. The necessary and sufficient conditions for the existence of an asymptotically stable control law of the form of Eq. (20) are that the continuous-time systems, Eq. (14), and discrete-time systems, Eq. (16), be stabilizable and that the corresponding open-loop transfer matrices $[\mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}]$ and $[\mathbf{C}(z\mathbf{I}_n - \mathbf{F})^{-1}\mathbf{G}]$ have no zeros at the origin and at $z = 1$, respectively. It should also be clear that if a constant reference input $\mathbf{y}_d$ is to be included in the problem, the error is $\mathbf{y} - \mathbf{y}_d$ and should be used instead of $\mathbf{y}$ in Eqs. (18) and (19). In this case, the error will go to zero in the steady state while the output $\mathbf{y}$ reaches $\mathbf{y}_d$. The advantage of the integral control action is that it reduces the steady-state error to zero without requiring knowledge of the constant disturbance input or accurate values of the system parameters as in Eqs. (11)–(13). The disadvantage is that it increases the order of the system and, in practice, would degrade system stability or speed of response.

3 THE STANDARD LINEAR QUADRATIC REGULATOR PROBLEM

Controller design in regulation applications using pole placement specifications emphasizes only the transient behavior of the state variables as the system returns to equilibrium. There is no explicit consideration of the required control effort. Control effort can be considered if the controller design problem is formulated as an optimal-control problem with weighting of both control effort and state-variable transients. For regulation applications, the index of

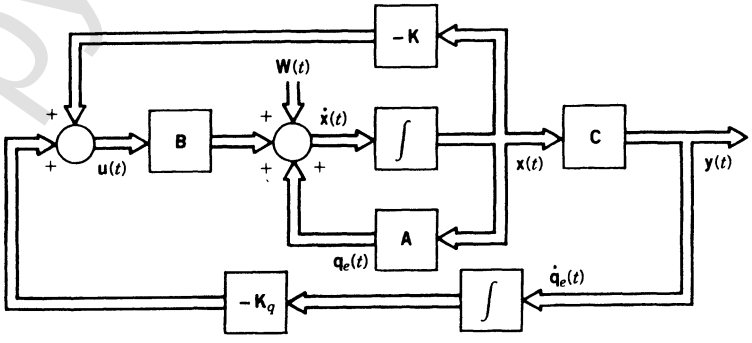


Figure 3 Proportional-plus-integral control, continuous-time system.

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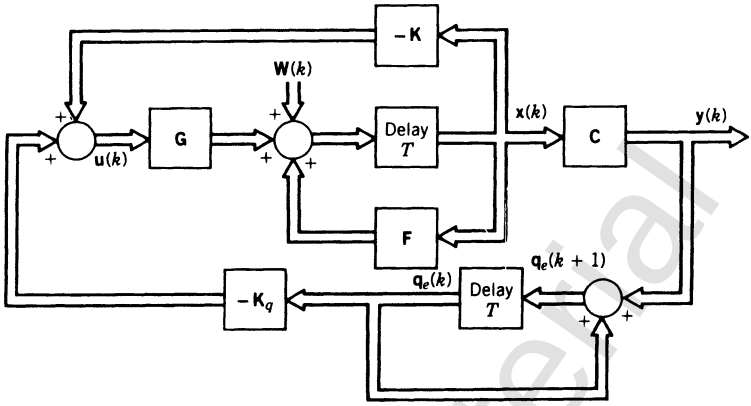


Figure 4 Proportional-plus-integral control, discrete-time system.

performance to be optimized, J, is usually chosen to be a quadratic function of the control inputs and the state variables. The resulting optimal-control law for the control input, when expressed in feedback form, is a linear function of the system state. Hence, this approach to control system design is referred to as the linear quadratic regulator (LQR) problem.

3.1 The Continuous-Time LQR Problem

Consider the continuous-time, LTV system described by Eq. (4) in Chapter 17 and the initial condition, for regulation problems, of

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{23}$$

The controller design problem is to choose the control input $\mathbf{u}(t)$ to minimize the quadratic index of performance[3]:

$$J = \int_{t_0}^{t_1} [\mathbf{x}^{\mathrm{T}}(t)\mathbf{R}_1(t)\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}_2(t)\mathbf{u}(t)]\, dt + \mathbf{x}^{\mathrm{T}}(t_1)\mathbf{P}_f\mathbf{x}(t_1) \tag{24}$$

$\mathbf{R}_1(t)$ is a positive-semidefinite symmetric weighting matrix on the state variables, $\mathbf{R}_2(t)$ is a positive-definite symmetric weighting matrix on the control inputs, and $\mathbf{P}_f$ is a positive-semidefinite symmetric weighting matrix on the terminal state. Times t_0, t_1 are initial and terminal time instants. The problem, as previously formulated, is a finite-time, deterministic, LQR problem. Kwakernaak and Sivan[3] have also considered a more general J that includes an additional term of the form $\mathbf{x}^{\mathrm{T}}(t)\mathbf{R}_{12}(t)\mathbf{u}(t)$ within the integral. They have shown that this J can be reduced to the form of Eq. (24) by appropriate redefinition of the weighting matrices and control vector.

The solution of this control problem, using methods from calculus of variations, can be obtained from standard textbooks on optimal control,[3,10] along with conditions for its existence and uniqueness. The optimal-control law, given in feedback form, is a linear, time-varying function of the system state:

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)\mathbf{x}(t) \tag{25}$$

where $\mathbf{P}$(t) is a $n \times n$ symmetric positive-semidefinite matrix satisfying the matrix Riccati equation

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$$-\dot{\mathbf{P}}(t) = \mathbf{R}_1(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}_2^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) \tag{26}$$

and the terminal condition

$$\mathbf{P}(t_1) = \mathbf{P}_f \tag{27}$$

Numerical solution of the matrix Riccati equation is a subject of great importance and of considerable research. Some useful techniques have been briefly described by Kwakernaak and Sivan.[3]

Solution of the LQR problem simplifies as the terminal time t_1 approaches infinity. It can be shown then that the solution of the matrix Riccati equation approaches a steady-state solution $\mathbf{P}_s(t)$ that is independent of $\mathbf{P}_f$. The resulting steady-state control law

$$\mathbf{u}(t) = -\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}_s(t)\mathbf{x}(t) \tag{28}$$

results in an exponentially stable closed-loop system if:

1. The linear system of Eq. (4) in Chapter 17 is uniformly completely state controllable.
2. The pair $\mathbf{A}(t)$, $\mathbf{H}_r^{\mathrm{T}}(t)$ is uniformly completely reconstructible where $\mathbf{H}_r(t)$ is any matrix such that $\mathbf{H}_r(t)\mathbf{H}_r^{\mathrm{T}}(t)$ equals $\mathbf{R}_1(t)$.

The matrix Riccati equation for the steady-state LQR problem simplifies to an algebraic equation and $\mathbf{P}_s$ is a constant if the system matrices and the weighting matrices in the index of performance are constant. The resulting algebraic Riccati equation is

$$\mathbf{R}_1 - \mathbf{P}_s\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}_s + \mathbf{A}^{\mathrm{T}}\mathbf{P}_s + \mathbf{P}_s\mathbf{A} = 0 \tag{29}$$

where $\mathbf{P}_s$ is a unique positive-definite solution of Eq. (29) and the resulting time-invariant closed-loop system is asymptotically stable if:

1. The linear system of Eq. (6) in Chapter 17 is completely state controllable.
2. The pair $\mathbf{A}$, $\mathbf{H}_r^{\mathrm{T}}$ is completely observable (reconstructible), where $\mathbf{H}_r$ is any matrix such that $\mathbf{H}_r\mathbf{H}_r^{\mathrm{T}}$ equals $\mathbf{R}_1$.

Another version of the LQR problem involves minimization of the quadratic index of performance for an LTI system over a finite time interval. If the weighting matrices are also time invariant, in many cases the optimal feedback gains are constant over most of the time interval of interest and vary with time only near the terminal time. Since constant feedback gains are easier to implement in practice, implementation of constant gains over the entire time interval would represent a nearly optimal solution that is practically more convenient.[3]

3.2 The Discrete-Time LQR Problem

The results of the LQR problem for discrete-time systems parallel those for continuous-time systems already stated. They are summarized here and described in greater length by Kwakernaak and Sivan.[3] The time-varying, discrete-time system is described by Eq. (10) in Chapter 17 and the initial condition

$$\mathbf{x}(k_0) = \mathbf{x}_0 \tag{30}$$

The index of performance to be minimized by controller design, for the finite-time LQR problem, is

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$$J = \sum_{k=k_0}^{k_1-1} [(\mathbf{x}^{\mathrm{T}})^{-1}(k+1)\mathbf{R}_1(k+1)\mathbf{x}(k+1) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{R}_2(k)\mathbf{u}(k)] + \mathbf{x}^{\mathrm{T}}(k_1)\mathbf{P}_f\mathbf{x}(k_1) \tag{31}$$

where the weighting matrices $\mathbf{R}_1(k)$, $\mathbf{R}_2(k)$, and $\mathbf{P}_f$ serve the same functions as $\mathbf{R}_1(t)$, $\mathbf{R}_2(t)$, and $\mathbf{P}_f$ for continuous-time systems and satisfy the same conditions. The values k_0, k_1 are the initial and final time instants. A more general version of the J, including the term $\mathbf{x}^{\mathrm{T}}(k)\mathbf{R}_{12}(k)\mathbf{u}(k)$ within the summation sign, can be reduced to the form of Eq. (31) by appropriate redefinition of the weighting matrices and control vector.[3]

The solution of this control problem can be obtained using dynamic programming methods and is given by

$$\mathbf{u}(k) = -\mathbf{K}(k)\mathbf{x}(k) \tag{32}$$

where

$$\mathbf{K}(k) = \{\mathbf{R}_2(k) + \mathbf{G}^{\mathrm{T}}(k)[\mathbf{R}_1(k+1) + \mathbf{P}(k+1)]\mathbf{G}(k)\}^{-1} \times \mathbf{G}^{\mathrm{T}}(k)[\mathbf{R}_1(k+1) + \mathbf{P}(k+1)]\mathbf{F}(k) \tag{33}$$

$\mathbf{P}(k)$ is a $n \times n$ symmetric, positive-semidefinite matrix satisfying the matrix difference equation

$$\mathbf{P}(k) = \mathbf{F}^{\mathrm{T}}(k)[\mathbf{R}_1(k+1) + \mathbf{P}(k+1)][\mathbf{F}(k) - \mathbf{G}(k)\mathbf{K}(k)] \qquad k = k_0, k_1 - 1 \tag{34}$$

with the terminal condition

$$\mathbf{P}(k_1) = \mathbf{P}_f \tag{35}$$

Unlike the matrix Riccati equation (26) for continuous-time systems, numerical solution of the preceding matrix difference equations is straightforward for finite-time LQR problems. The procedure involves solution of the difference equations backward in time:

1. Let $k = k_1 - 1$. Then $\mathbf{P}(k+1)$ is equal to $\mathbf{P}_f$ and hence is known.
2. Compute $\mathbf{K}(k)$ using Eq. (33) and the known value of $\mathbf{P}(k+1)$.
3. Compute $\mathbf{P}(k)$ using Eq. (34) and the known values of $\mathbf{K}(k)$ and $\mathbf{P}(k+1)$.
4. Reduce k by 1 and repeat 2 and 3 until $k = k_0$.

The solution to the discrete-time LQR problem also simplifies as the terminal time k_1 approaches infinity. The solutions of the matrix difference equations (33) and (34) converge to steady-state solutions $\mathbf{K}_s(k)$, $\mathbf{P}_s(k)$, which are independent of $\mathbf{P}_f$. The resulting steady-state control law

$$\mathbf{u}(k) = -\mathbf{K}_s(k)\mathbf{x}(k) \tag{36}$$

results in an exponentially stable closed-loop system if:

1. The linear system of Eq. (10) in Chapter 17 is uniformly completely state controllable.
2. The pair $\mathbf{F}(k)$, $\mathbf{H}_r^{\mathrm{T}}(k)$ is uniformly completely reconstructible where $\mathbf{H}_r(k)$ is any matrix such that $\mathbf{H}_r(k)\mathbf{H}_r^{\mathrm{T}}(k)$ equals $\mathbf{R}_1(k)$.

Also, the matrices $\mathbf{K}_s$ and $\mathbf{P}_s$ are constants if the system matrices and the weighting matrices in the index of performance of Eq. (31) are constants. They are given by solution of the following algebraic equations:

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$$\mathbf{K}_s = [\mathbf{R}_2 + \mathbf{G}^T(\mathbf{R}_1 + \mathbf{P}_s)\mathbf{G}]^{-1}\mathbf{G}^T(\mathbf{R}_1 + \mathbf{P}_s)\mathbf{F} \tag{37}$$

$$\mathbf{P}_s = \mathbf{F}^T(\mathbf{R}_1 + \mathbf{P}_s)(\mathbf{F} - \mathbf{G}\mathbf{K}_s) \tag{38}$$

The optimal-control law for the infinite-time LQR problem is

$$\mathbf{u}(k) = -\mathbf{K}_s\mathbf{x}(k) \tag{39}$$

and requires only constant state feedback gains. The solution $\mathbf{P}_s$ of Eqs. (37) and (38) is positive definite, and the optimal-control law results in an asymptotically stable closed-loop system if:

1. The linear system of Eq. (12) in Chapter 17 is completely state controllable.
2. The pair $\mathbf{F}$, $\mathbf{H}_r^T$ is completely observable (reconstructible), where $\mathbf{H}_r$ is any matrix such that $\mathbf{H}_r\mathbf{H}_r^T$ equals $\mathbf{R}_1$.

Also, as in the case of continuous-time systems, the optimal feedback gains are nearly constant even for finite-time LQR problems if the system matrices and weighting matrices in J are constant.[11] Finally, a number of techniques for solving the matrix algebraic equations (37) and (38) and the matrix difference equations (33)–(35) are described by Kuo.[12]

3.3 Stability and Robustness of the Optimal-Control Law

An important consideration in the practical usefulness of the optimal-control laws for the LQR problems described is the implication of these laws for performance features of the controlled systems not included in J, such as relative stability and sensitivity of the controlled system to unmodeled dynamics or plant parameter variations. Reference has already been made to the fact that the optimal-control laws for the continuous-time and discrete-time infinite-time LQR problems described result in asymptotically stable closed-loop systems provided that specified controllability and reconstructibility or observability conditions are satisfied. Closed-loop systems with a prescribed degree of stability can be obtained by modifying the performance index J for linear, time-invariant, continuous-time systems:[10]

$$J = \int_0^\infty e^{2\alpha t}(\mathbf{x}^T\mathbf{R}_1\mathbf{x} + \mathbf{u}^T\mathbf{R}_2\mathbf{u})dt \tag{40}$$

where α is a positive scalar constant. If the pair $\mathbf{A}$, $\mathbf{B}$ is completely state controllable and the pair $\mathbf{A}$, $\mathbf{H}_r^T$ is completely observable where $\mathbf{H}_r\mathbf{H}_r^T$ is equal to $\mathbf{R}_1$, the solution to this LQR problem results in a finite value of J. Hence, the transients decay at least as rapidly as $e^{-\alpha t}$. Larger values of α would therefore ensure a more rapid return of the system to equilibrium. The corresponding algebraic Riccati equation is

$$\mathbf{R}_1 - \mathbf{P}_s\mathbf{B}\mathbf{R}_2^{-1}\mathbf{B}^T\mathbf{P}_s + \mathbf{A}^T\mathbf{P}_s + \mathbf{P}_s\mathbf{A} + 2\alpha\mathbf{P}_s = 0 \tag{41}$$

and the optimal-feedback-control law is given by Eq. (28). A similar procedure for discrete-time LTI systems is described by Franklin and Powell.[11]

Additional results concerning the stability properties of the optimal control law for continuous-time, LTI systems described by Eq. (6) in Chapter 17 and employing only constant weighting matrices in the index of performance, Eq. (24), are available and will be summarized. Anderson and Moore[10] have shown that for single-input systems the optimal-control law for the infinite-time LQR problem has $\pm 60°$ phase margin, an infinite gain margin, and 50% gain reduction tolerance before the closed-loop system becomes unstable. These results are best explained with the aid of Fig. 5*a*, where $G_p(s)$ is normalized to be

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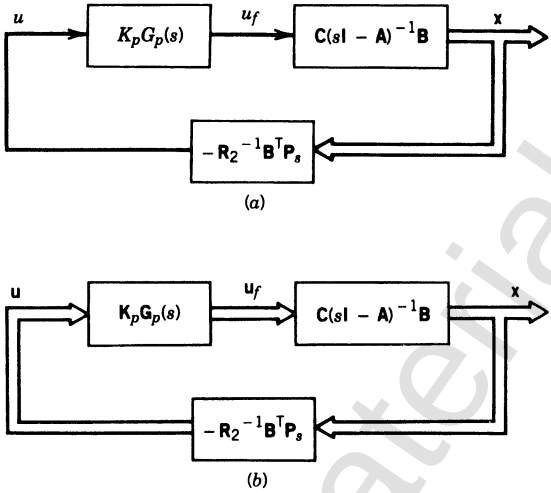


Figure 5 Robustness of optimal LQR control: (*a*) single-input system and (*b*) multi-input system.

unity at $s = 0$. The transfer function $K_pG_p(s)$ characterizes the modeling accuracy and is unity for an exact model. The result stated previously indicates that modeling errors that result either in phase shifts $\underline{/G_p(j\omega)}$ of less than 60° in magnitude for all frequencies or in values of the magnitude ratio K_p greater than one-half would not destabilize the closed-loop system. The result on gain margins extends also to static nonlinear gain relationships between u_f and u.[10]

Safonov[13] has extended these results on gain and phase margins to multi-input infinite-time LQR problems. As shown in Fig. *5b,* the quantities $\mathbf{u}_f(s)$ and $\mathbf{u}(s)$ are vectors and the modeling accuracy, if linear and time invariant, is represented by a transfer matrix $\mathbf{K}_p\mathbf{G}_p^{(s)}$. For an exact model, $\mathbf{K}_p\mathbf{G}_p(s)$ is the identity matrix. The results stated here are special cases of the results derived by Safonov[13] and are valid for the case where $\mathbf{K}_p$ and $\mathbf{G}_p(s)$ are diagonal matrices and $\mathbf{G}_p(s)$ consists of normalized transfer functions $\mathbf{G}_{pj}(s)$ [i.e., $\mathbf{G}_p(0)$ is the identity matrix]. For such a case, as long as all of the phase shifts $\underline{/G_{pj}(j\omega)}$ are less than 60° in magnitude for all frequencies or as long as all of the elements of the $\mathbf{K}_p$ matrix are greater than one-half, the closed-loop system using the optimal-control law is asymptotically stable. More general robustness results, which are more difficult to apply, are also available.[13,14]

Perkins and Cruz[15] have examined frequency-domain characterizations of the infinite-time LQR problem for continuous-time LTI systems and have shown that feedback realization of the optimal-control law results in lower sensitivity to plant parameter variations than an equivalent open-loop realization. Kwakernaak and Sivan[3] have provided other results that are somewhat more useful in relating the sensitivity properties of optimal-control laws to the weighting matrices in J. As the elements of the control effort weighting matrix $\mathbf{R}_2$ are decreased, the control law sensitivity decreases or improves since the optimal feedback gains are higher. However, higher feedback gains naturally imply greater likelihood of actuator saturation. The relative sensitivities of the different state variables depend on the elements of the state weighting matrix $\mathbf{R}_1$. State variables that are weighted more heavily would have lower sensitivity. Also, the sensitivity characteristics of the optimal-control law for non-minimum-phase systems are shown to be inferior to that of minimum-phase systems. Finally, Kwakernaak and Sivan[3] have illustrated that the sensitivity results described do not necessarily extend to discrete-time systems.

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4 EXTENSIONS OF THE LINEAR QUADRATIC REGULATOR PROBLEM

The optimal-control law for the LQR problem and the pole placement design method described in the preceding sections have a number of limitations. First, as Horowitz[16] has pointed out, control system design by pole placement or quadratic performance index minimization obscures some practically important aspects and objectives. Among these are sensor noise, loop bandwidths, and sensitivity of system performance to significant plant parameter variations. Rosenbrock and McMorran[17] have pointed out that unconditional stability (i.e., stability for all values of the control gains between zero and design values) is essential for industrial control systems but is not guaranteed by the optimal-control law. Hence, for multivariable optimal-control systems, the failure of a single feedback-measuring instrument could destabilize the closed-loop system. Moreover, the achievement of more modest sensitivity requirements is complicated by the lack of clear guidelines for weighting matrix selection. Available procedures for weighting matrix selection enable the achievement of desired transient response characteristics either by specification of a few dominant closed-loop system poles[10] or by implicit model reference following methods.[18] In the latter case, the reference model is chosen to have desired transient response characteristics. However, there is no available method to ensure a priori that the optimal-control law has other desirable performance characteristics such as low sensitivity. The consequence of the lack of satisfactory guidelines for weighting matrix selection is that practical control system design using the LQR formulation involves considerable trial and error.

Second, the formulation of the standard LQR problem needs to be extended to be able to effectively handle control problems other than regulation. Examples of such problems include regulation in the presence of persistent disturbances, tracking problems, and vibration control problems. Even though some of these problems can be handled by simple extensions of the LQR problem, effective solutions to these problems require significant extensions of the LQR problem formulation.

Extensions of the standard LQR problem formulation addressing some of its limitations will be described here. The extensions involve alternative formulations of the quadratic index of performance to be minimized such that the resulting solutions have desired features. Additionally, the problem formulation utilizes more completely the available information on the systems to be controlled and their environments. One of the measures for evaluating the effectiveness of the resulting problem formulations and solutions is their ability to accommodate a greater variety of problems and performance specifications. Another such measure is their ability to incorporate in the proposed solutions features that are known to be effective in practice. The resulting variety of problem formulations and solutions runs somewhat counter to the unifying nature of the standard LQR problem formulation and constitutes a recognition of its limitations in practice.

4.1 Disturbance Accommodation

Extensions of the standard LQR problem to accommodate unknown disturbance inputs have been proposed by Anderson and Moore,[10] Johnson,[19,20] and Davison and Ferguson.[21] Anderson and Moore[10] consider LTI systems of the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}[\mathbf{u}(t) + \mathbf{M}_w\mathbf{w}] \tag{42}$$

where $\mathbf{w}$ is a constant, unknown disturbance vector. The restriction that $\mathbf{u}(t)$ and $\mathbf{M}_w\mathbf{w}$ occur additively ensures that the equilibrium $\mathbf{x} = 0$ can be achieved and maintained even if $\mathbf{w}$ is nonzero. The following index of performance with input derivative constraints is to be minimized by the control law:

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$$J = \int_0^\infty [\mathbf{x}^T\mathbf{R}_1\mathbf{x} + (\mathbf{u} + \mathbf{M}_w\mathbf{w})^T\mathbf{R}_2(\mathbf{u} + \mathbf{M}_w\mathbf{w}) + \dot{\mathbf{u}}^T\mathbf{R}_3\dot{\mathbf{u}}]\, dt \tag{43}$$

where $\mathbf{R}_1$, $\mathbf{R}_2$ are symmetric positive-semidefinite matrices and $\mathbf{R}_3$ is a symmetric positive-definite matrix. When the system state is augmented to include the vector $\mathbf{u} + \mathbf{M}_w\mathbf{w}$ and the input derivative $\dot{\mathbf{u}}$ is defined to be the new input, the problem reduces to the standard infinite-time LQR problem. The optimal-control law is a proportional-plus-integral state feedback law:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) - \mathbf{K}_I \int \mathbf{x}\, dt \tag{44}$$

where the constant gain matrices $\mathbf{K}$, $\mathbf{K}_I$ are known linear functions of matrices satisfying algebraic Riccati equations. In cases where the complete state $\mathbf{x}$ is not measurable, the state would be estimated from the measured outputs using the methods described in Section 5. The closed-loop system is asymptotically stable provided that certain controllability and observability conditions on the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{R}_1$, and $\mathbf{R}_2$ are satisfied. The gain and phase margin results noted earlier for the standard infinite-time LQR problem are valid here as well.

Johnson[19] considers a more general class of disturbances and state-space equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{B}_w(t)\mathbf{w}(t) \tag{45}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) + \mathbf{D}_w(t)\mathbf{w}(t) \tag{46}$$

where $\mathbf{w}(t)$ is a vector of disturbance inputs. The disturbance inputs are assumed to be described by linear time-varying differential equations that constitute a state-space model for the disturbances:

$$\dot{\mathbf{z}}_\sigma(t) = \mathbf{A}_\sigma(t)\mathbf{z}_\sigma(t) + \mathbf{B}_\sigma(t)\mathbf{x}(t) + \sigma(t) \tag{47}$$

$$\mathbf{w}(t) = \mathbf{C}_\sigma(t)\mathbf{z}_\sigma(t) + \mathbf{D}_\sigma(t)\mathbf{x}(t) \tag{48}$$

where $\mathbf{z}_\sigma(t)$ represents the state of the disturbance and $\boldsymbol{\sigma}(t)$ is a vector of Dirac delta impulses occurring at unknown times. The terms including $\mathbf{x}(t)$ in the preceding equations enable cases of state-dependent disturbances to be considered within this framework. The coefficient matrices are determined experimentally by examination of the records of the disturbances. This type of description of disturbances constitutes a waveform mode description and is applicable to a broad class of disturbances of practical interest that are not described well either by deterministic process models or by stochastic process models. Examples of such disturbances are piecewise linear, piecewise polynomial, or piecewise periodic signals.

The waveform mode description of disturbances is combined with the system state-space equations to provide a rather complete description of the system to be controlled and the inputs affecting its behavior. The exact design of the controller depends on the specific objectives used to govern the design. If one of the objectives of the control system design is to counteract as completely as possible the effects of the disturbance inputs, then the control input is considered to be composed of two parts:

$$\mathbf{u}(t) = \mathbf{u}_m(t) + \mathbf{u}_d(t) \tag{49}$$

where $\mathbf{u}_d(t)$ is the component used to counteract disturbance effects either completely or partially and $\mathbf{u}_m(t)$ is the component used to accomplish other objectives such as closed-loop pole placement. Alternatively, the objective of control system design may be the minimization of a quadratic index of performance. In either of these cases, the control law would

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require the feedback of the system state $\mathbf{x}$ as well as the disturbance state $\mathbf{z}_\sigma$ (Fig. 6). The state estimation methods described in Section 5 can be used to generate these state estimates from available measurements. The extension of this disturbance accommodation approach to discrete-time systems has also been considered by Johnson.[20]

The formulation of disturbance state models and their incorporation in controller design result in controller features familiar from more classical approaches to disturbance suppression. Examples are integral control for constant disturbances, notch filter control for sinusoidal disturbances, and disturbance feedforward if some components of the disturbance inputs are measurable.[19] Hence, the disturbance accommodation controllers described here may be viewed as generalizations of classical solutions to disturbance suppression. A similar comment may be made concerning the mechanism for disturbance suppression inherent in the robust servomechanism structure described by Davison and Ferguson.[21] The robust controller structure is described later in this section.

4.2 Tracking Applications

Anderson and Moore,[10] Davison and Ferguson,[21] Trankle and Bryson,[22] and Tomizuka et al.[23–26] have considered extensions of the LQR formulation to accommodate tracking applications. Anderson and Moore[10] have considered the servomechanism problem where the linear system state equations are given by Eqs. (4) and (5) in Chapter 17 with $\mathbf{D}(t) = \mathbf{0}$ and the class of desired trajectories $\mathbf{y}_r$ is given by

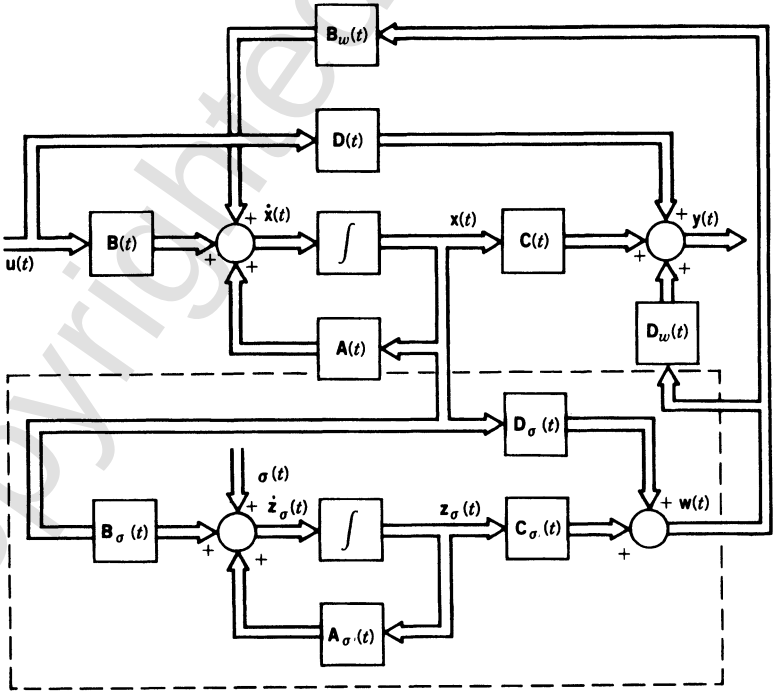


Figure 6 Disturbance state model, continuous-time system.

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$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r(t)\mathbf{x}_r(t) \tag{50}$$

$$\mathbf{y}_r(t) = \mathbf{C}_r(t)\mathbf{x}_r(t) \tag{51}$$

and a specified initial condition $\mathbf{x}_r(t_0)$. The index of performance to be optimized for the finite-time problem is

$$J = \int_{t_0}^{t_1} \{\mathbf{x}^{\mathrm{T}}[\mathbf{I} - \mathbf{C}^{\mathrm{T}}(\mathbf{C}\mathbf{C}^{\mathrm{T}})^{-1}\mathbf{C}]^{\mathrm{T}}\mathbf{Q}_1[\mathbf{I} - \mathbf{C}^{\mathrm{T}}(\mathbf{C}\mathbf{C}^{\mathrm{T}})^{-1}\mathbf{C}]\mathbf{x} + (\mathbf{y} - \mathbf{y}_r)^{\mathrm{T}}\mathbf{Q}_2(\mathbf{y} - \mathbf{y}_r) + \mathbf{u}^{\mathrm{T}}\mathbf{R}_2\mathbf{u}\}dt \tag{52}$$

where the time dependencies of the vectors and matrices have been omitted for convenience. The matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are positive-semidefinite matrices and $\mathbf{R}_2$ is a positive-definite matrix. The weighting on the tracking error $\mathbf{y} - \mathbf{y}_r$ helps reduce it, whereas the weighting on the state $\mathbf{x}$ achieves a smooth response. The optimal-control law involves linear feedback of the system state as well as feedforward of the state of the trajectory model:

$$\mathbf{u} = -\mathbf{K}(t)\mathbf{x}(t) - \mathbf{K}_r(t)\mathbf{x}_r(t) \tag{53}$$

where the gain matrices $\mathbf{K}(t)$ and $\mathbf{K}_r$ are linearly related to solutions of the matrix Riccati differential equations. Conditions for the time-invariant version of this servo problem to reduce to the standard infinite-time LQR problem have also been noted.[10]

Trankle and Bryson[22] have considered the time-invariant servomechanism problem for the case where $\mathbf{y}$, $\mathbf{y}_r$, $\mathbf{u}$ have the same dimension and have proposed the following index of performance:

$$J = \int_0^{\infty} [(\mathbf{y} - \mathbf{y}_r)^{\mathrm{T}}\mathbf{Q}_y(\mathbf{y} - \mathbf{y}_r) + (\mathbf{u} - \mathbf{U}_1\mathbf{x}_r)^{\mathrm{T}}\mathbf{R}_u(\mathbf{u} - \mathbf{U}_1\mathbf{x}_r)]\, dt \tag{54}$$

where $\mathbf{Q}_y$ is positive semidefinite and $\mathbf{R}_u$ is positive definite. A modification of the index of performance to add integral error feedback can also be devised. A matrix $\mathbf{U}_1$ and another matrix $\mathbf{X}$ to occur later in the development are defined by

$$\mathbf{C}\mathbf{X} = \mathbf{C}_r \tag{55}$$

$$\mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U}_1 = \mathbf{X}\mathbf{A}_r \tag{56}$$

The optimal-control law is asymptotically stable if the pair $\mathbf{A}$, $\mathbf{B}$ is completely state controllable and the pair $\mathbf{A}$, $\mathbf{C}$ is completely observable. The control law is given by

$$\mathbf{u} = (\mathbf{U}_1 + \mathbf{K}\mathbf{X})\mathbf{x}_r(t) - \mathbf{K}\mathbf{x}(t) \tag{57}$$

where $\mathbf{K}$ is related, in the usual manner, to the solution of an algebraic Riccati equation. The first term on the right-hand side represents feedforward control action, and the second term represents feedback control action (Fig. 7). The feedforward action yields faster and more accurate tracking of the desired trajectory than other control schemes that depend more on integral error feedback. Finally, model and system state feedback is required by both Eqs. (53) and (57). If these states are not available for measurement, state estimators such as those described in Section 5 would be needed.

A variation on the servomechanism problem already described is that of tracking where the desired trajectory $\mathbf{y}_r$ is known a priori rather than being defined by a model as in Eqs. (50) and (51). Anderson and Moore[10] have determined the optimal-control law for the index of performance Eq. (52):

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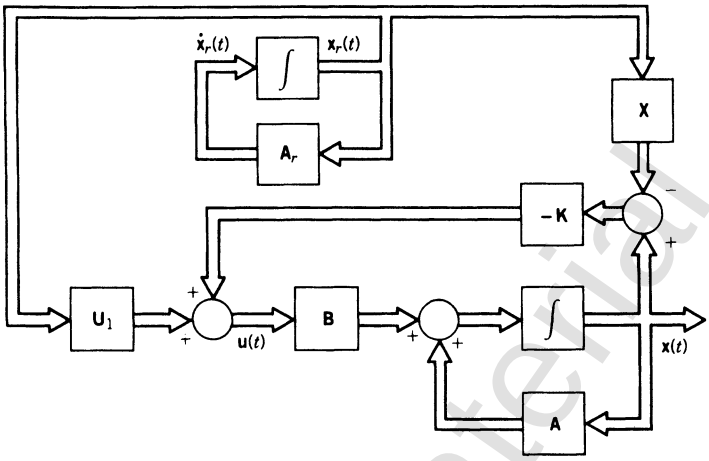


Figure 7 Extension of LAR for time-invariant servomechanism problem.

$$u = -\mathbf{K}(t)\mathbf{x}(t) - \mathbf{R}_2(t)^{-1}\mathbf{B}^{\mathrm{T}}(t)\mathbf{b}(t) \tag{58}$$

where $\mathbf{b}(t)$ is the solution of a linear ordinary differential equation with $\mathbf{y}_r(t)$ as the forcing function and $\mathbf{K}(t)$ is related, in the usual manner, to the solution of a matrix Riccati differential equation.

The control law, Eq. (58), incorporates information about future inputs over the entire interval of interest (t_0, t_1) and is said to have infinite preview control in addition to feedback control. A related problem is one where only finite preview of the desired trajectory is available; that is, at any time τ, $\mathbf{y}_r(t)$ is known for $\tau \leq t \leq \Delta T + \tau$, where ΔT is called the preview length. Tomizuka[23] has examined the continuous-time finite preview problem and determined the optimal-control law for a quadratic index of performance over the entire interval of interest (t_0, t_1) where t_1 is greater than $t_0 + \Delta T$. The desired trajectory, not known from preview at any time t, is assumed to be modeled by a stochastic process. A discrete-time version of the problem is given by Tomizuka and Whitney.[24] Discrete-time finite preview of disturbance inputs in addition to the desired trajectory has also been considered by Tomizuka et al.[25,26] The results indicate that preview control improves the control system performance, especially in the low-frequency range, and that there exists a critical preview length beyond which preview information is less important.

4.3 Frequency Shaping of Cost Functionals

Extensions of the standard LQR problem and the resulting control laws have certain limitations related to the nature of the index of performance used. These limitations become clear when the index of performance is viewed in the frequency domain.[27] The index of performance for the infinite-time, LQR problem for an LTI system

$$J = \int_0^\infty (\mathbf{x}^{\mathrm{T}}\mathbf{R}_1\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}_2\mathbf{u})\, dt \tag{59}$$

can be transformed to the frequency domain using Parseval's theorem:

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$$J = \frac{1}{2}\int_{-\infty}^{\infty} [\mathbf{x}^*(j\omega)\mathbf{R}_1\mathbf{x}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{R}_2\mathbf{u}(j\omega)]\, d\omega \tag{60}$$

where the asterisk implies a complex-conjugate transpose of a complex matrix. The index of performance thus weights control and state transients at all frequencies equally despite the fact that, in practice, model accuracy is poorer at high frequencies. Model inaccuracy, sensor noise, and actuator bandwidth limitations are better accommodated by an index of performance that penalizes high-frequency control activity more heavily than low-frequency control activity. In fact, it is recognized good practice in classical control system design to have a steep enough rolloff or attenuation rate of the open-loop transmission functions at high frequencies for adequate noise suppression and robustness to model errors. In contrast, the closed-loop amplitude ratio frequency response curve corresponding to the optimal-control law for an LQR problem may drop off only as slowly as 20 dB/decade.[27] Another limitation of the standard LQR problem formulation is that, in many applications such as vibration control, the objectives of control are stated better in the frequency domain than in the time domain.

Specification of the index of performance in the frequency domain with the weighting matrices being functions of the frequency ω enables many of these limitations to be removed. The frequency-shaped cost functional method described by Gupta[27] is one such method and allows the generalized index of performance:

$$J = \frac{1}{2}\int_{-\infty}^{\infty} [\mathbf{x}^*(j\omega)\mathbf{R}_1(j\omega)\mathbf{x}(j\omega) + \mathbf{u}^*(j\omega)\mathbf{R}_2(j\omega)\mathbf{u}(j\omega)\, d\omega \tag{61}$$

The restrictions on the weighting matrices are:

1. $\mathbf{R}_1(j\omega)$ and $\mathbf{R}_2(j\omega)$ are positive semidefinite and positive definite, respectively, at all frequencies.
2. $\mathbf{R}_1(j\omega)$ and $\mathbf{R}_2(j\omega)$ are Hermitian matrices at all frequencies and, in fact, are rational functions of ω^2
3. $\mathbf{R}_2(j\omega)$ has rank r, where r is the dimension of control vector $\mathbf{u}$.

Two new matrices $\mathbf{P}_1(j\omega)$ and $\mathbf{P}_2(j\omega)$ are defined based on $\mathbf{R}_1(j\omega)$ and $\mathbf{R}_2(j\omega)$:

$$\mathbf{R}_1(j\omega) = \mathbf{P}_1^*(j\omega)\mathbf{P}_1(j\omega) \tag{62}$$

$$\mathbf{R}_2(j\omega) = \mathbf{P}_2^*(j\omega)\mathbf{P}_2(j\omega) \tag{63}$$

where $\mathbf{P}_1$ is $m \times n$, m being the rank of $\mathbf{R}_1$, and $\mathbf{P}_2$ is $r \times r$, r being the rank of $\mathbf{R}_2$ and the dimension of the control vector $\mathbf{u}$. New vectors $\mathbf{x}_p$, and $\mathbf{u}_p$, dynamically related to $\mathbf{x}$ and $\mathbf{u}$, respectively, are then defined:

$$\mathbf{x}_p(s) = \mathbf{P}_1(s)\mathbf{x}(s) \tag{64}$$

$$\mathbf{u}_p(s) = \mathbf{P}_2(s)\mathbf{u}(s) \tag{65}$$

Minimal realizations of $\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$ are then determined as described in Chapter 18, Section 7, and new states $\mathbf{z}_x$ and $\mathbf{z}_u$ defined:

$$\dot{\mathbf{z}}_x = \mathbf{A}_x\mathbf{z}_x + \mathbf{B}_x\mathbf{x} \tag{66}$$

$$\mathbf{x}_p = \mathbf{C}_x\mathbf{z}_x + \mathbf{D}_x\mathbf{x} \tag{67}$$

$$\dot{\mathbf{z}}_u = \mathbf{A}_u\mathbf{z}_u + \mathbf{B}_u\mathbf{u} \tag{68}$$

$$\mathbf{u}_p = \mathbf{C}_u\mathbf{z}_u + \mathbf{D}_u\mathbf{u} \tag{69}$$

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An augmented state $\mathbf{x}_a$ is then defined:

$$\mathbf{x}_a = \begin{bmatrix} \mathbf{x} \\ \mathbf{z}_x \\ \mathbf{z}_u \end{bmatrix} \tag{70}$$

Using Parseval's theorem, the index of performance, Eq. (61), is transformed to the time domain to yield a standard LQR problem formulation with constant weighting matrices:

$$J = \int_0^\infty (\mathbf{x}_a^{\mathrm{T}}\mathbf{R}_{1a}\mathbf{x}_a + 2\mathbf{x}_a^{\mathrm{T}}\mathbf{R}_{12a}\mathbf{u} + \mathbf{u}^{\mathrm{T}}\mathbf{R}_{2a}\mathbf{u})\, dt \tag{71}$$

The optimal-control law is obtained by solving the corresponding algebraic matrix Riccati equation:

$$\mathbf{u} = -(\mathbf{K}\mathbf{x} + \mathbf{K}_x\mathbf{z}_x + \mathbf{K}_u\mathbf{z}_u) \tag{72}$$

The states $\mathbf{z}_x$ and $\mathbf{z}_u$, are dynamically related to $\mathbf{x}$ and $\mathbf{u}$, respectively. Hence, the optimal-control law has dynamic compensators in addition to linear instantaneous state-variable feedback (Fig. 8). If the state $\mathbf{x}$ is not completely measurable, state estimation is required as described in Section 5.

The utility of this design method is that it establishes a clear link between features of the weighting matrices $\mathbf{R}_1(j\omega)$ and $\mathbf{R}_2(j\omega)$ and the resulting controllers. Gupta[27] has shown that the compensator poles and zeros are the same as poles and zeros of the transfer functions $\mathbf{P}_1(s)$ and $\mathbf{P}_2(s)$. For example, if $\mathbf{R}_1(j\omega)$ is singular at $\omega = 0$, integral control results. If $\mathbf{R}_1(j\omega)$ is singular at any other frequency ω_1, the controller has a notch filter at that frequency. This would be desirable if the controlled system has a known resonant frequency at ω_1 and we wish to minimize the excitation of the resonance by disturbances. Finally, if $\mathbf{R}_2(j\omega)$ is chosen to increase with frequency ω, the optimal-control law would have reduced control action at high frequencies. As indicated previously, this is a desirable feature if the controller is to have good noise suppression and robustness to model errors.

4.4 Robust Servomechanism Control

Davison and Ferguson[21] have proposed a controller structure and formulated a design procedure for the robust control of servomechanisms. Robustness is defined here to imply asymptotic stability of the closed-loop system and asymptotic tracking of the desired trajectory for all initial conditions of the controller used and for all variations in the system model parameters that do not cause the controlled system to become unstable. The system equations are the time-invariant versions of Eqs. (45) and (46). The disturbance inputs $\mathbf{w}(t)$ are modeled by time-invariant versions of Eqs. (47) and (48) with no provision for either state-dependent disturbances [$\mathbf{B}_\sigma(t) = \mathbf{0} = \mathbf{D}_\sigma(t)$] or impulsive inputs [$\sigma(t) = 0$]. The desired trajectory $\mathbf{y}_r(t)$ is described by time-invariant versions of Eqs. (50) and (51).

Under certain specified conditions,[21] the robust servomechanism problem is assured of a solution. The resulting controller structure consists of a servocompensator and stabilizing compensator (Fig. 9), and the robust control input is given by

$$\mathbf{u} = -\mathbf{K}_\eta\eta - \mathbf{K}_\theta\theta \tag{73}$$

where η and θ are the outputs of the servocompensator and stabilizing compensator, respectively, and $\mathbf{K}_\eta$ and $\mathbf{K}_\theta$ are constant-gain matrices. The servocompensator is a dynamic controller with the trajectory error as input and its form and parameters are determined from the state-space models of the system and the disturbance and trajectory inputs. The servo-

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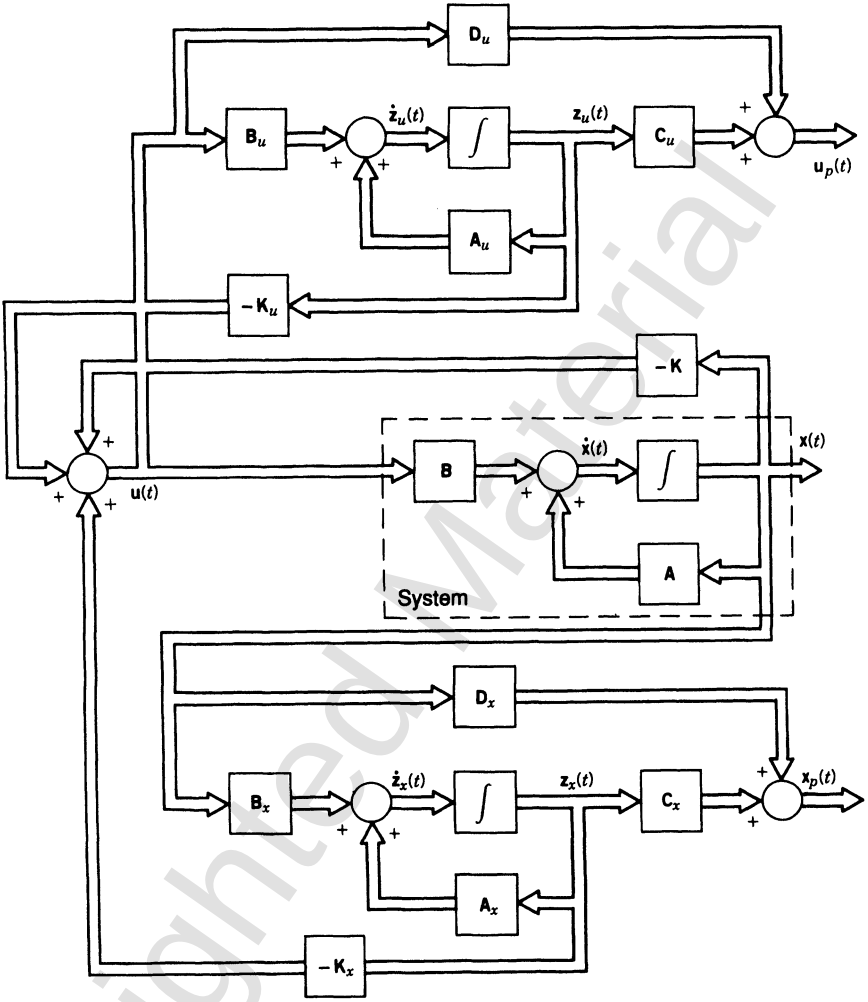


Figure 8 Extension of LQR using frequency-dependent cost functionals.

compensator is a generalization of the integral controller from classical control theory. The stabilizing compensator has the function of stabilizing the augmented system consisting of the servocompensator and the system to be controlled. In general, the stabilizing compensator has a number of inputs as shown in Fig. 9. It is not uniquely defined, however, and is usually chosen to have as simple a form as is possible given the performance requirements on the controlled system. Complete state feedback, if measurable, and observer-based controllers of the type described in Section 6 are among the more elaborate forms of the stabilizing compensator.

Once the structure of the stabilizing compensator is determined, the unknown controller parameters are determined by minimization of a quadratic index of performance. The index of performance is specified such that minimizing it gives a system with fast response and low interaction for MIMO systems. The optimum value of the index of performance is given

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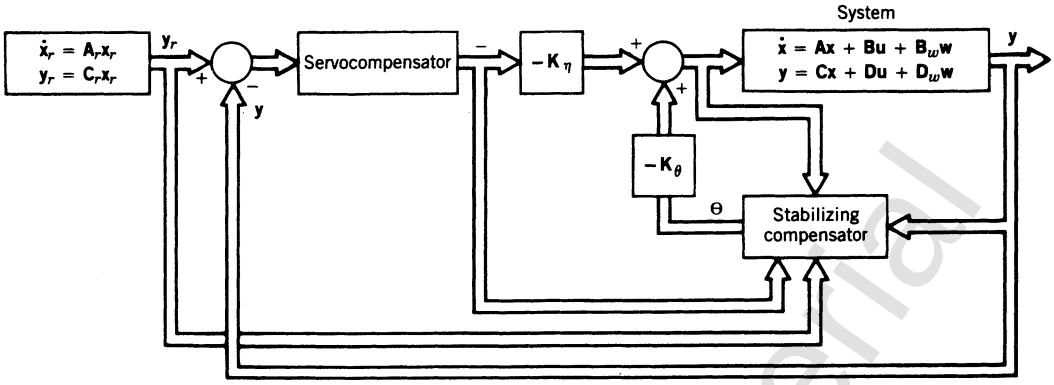


Figure 9 Extension of LQR for the robust LTI servomechanism problem.

in terms of the controller parameters. Since controller parameters are not known a priori, the optimal controller design reduces to a multiparameter optimization problem where the quantity being optimized is the quadratic index of performance. The parameter optimization can be constrained to allow the designer to handle closed-loop system damping constraints, controller gain constraints to avoid saturation effects, controller integrity constraints for sensor and actuator failures, and tolerance constraints to system parameter variations. When applied to example systems,[21] the robust control approach yields controller features commonly obtained from other frequency-domain-based design procedures, such as error integral control, phase-lead compensation, pole–zero cancellation, and low interaction for MIMO systems. Its ability to accommodate a variety of constraints on the controllers to ensure their practical utility makes the robust controller design approach a practically useful approach. Care is needed, however, to keep the resulting controller as simple as possible.

The recent extensions of the LQR problem described have addressed many of the limitations of state-space-based approaches to control system design noted by Horowitz[16] and others. As a result, the state-space approach is expected to be more useful in a greater variety of practical applications. It should be noted, however, that there are still no specific guidelines for the selection of weighting matrices in the quadratic indices of performance used by the extended versions of the LQR problem. Consequently, control system design using these methods would still involve considerable trial and error.

5 DESIGN OF LINEAR STATE ESTIMATORS

The optimal control laws for the standard LQR problem and the extensions described earlier require feedback of the entire system state. Pole placement algorithms require state feedback as well. Complete state measurement is not possible or practical in many instances. Therefore, the state variables often must be estimated from the measured output variables. Closed-loop dynamic realizations of such state estimators are described here. The property of reconstructibility (or equivalently, observability for LTI systems) is critical for the design of such estimators. It plays a role relative to state estimators that is very similar to the role played by state controllability relative to state feedback controller design. This similarity is the consequence of the duality of the two properties, referred to in Chapter 17, Section 6.

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Therefore, many of the results presented in this section parallel those of the preceding section on controller design.

5.1 The Observer

The structure of a closed-loop dynamic system, called an observer, to estimate the state $\mathbf{x}(t)$ of an LTV system described by Eqs. (4) and (5) from Chapter 17, from output measurements $\mathbf{y}(t)$ and input measurements $\mathbf{u}(t)$, was proposed by Luenberger[28] (Fig. 10):

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\,\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t) - \mathbf{D}(t)\mathbf{u}(t)] \tag{74}$$

where $\hat{\mathbf{x}}(t)$ is the estimated state and $\mathbf{L}(t)$ is a time-varying matrix of observer gains. Combination of Eq. (74) and Eqs. (4) and (5) from Chapter 17 yields a dynamic equation for the state estimation error $\mathbf{e}(t)$:

$$\dot{\mathbf{e}}(t) \stackrel{\Delta}{=} \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\mathbf{e}(t) \tag{75}$$

with the initial condition

$$\mathbf{e}(t_0) = \mathbf{x}(t_0) - \hat{\mathbf{x}}(t_0) \tag{76}$$

Thus, if the observer is asymptotically stable

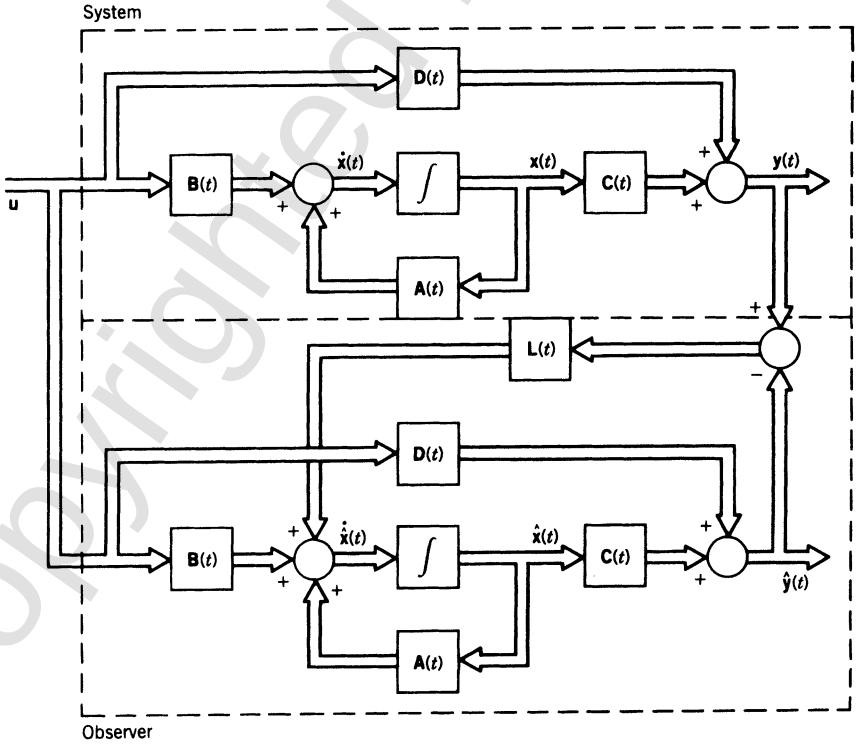


Figure 10 Observer for LTV continuous-time system.

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$$\lim_{t \to \infty} \mathbf{e}(t) = 0 \tag{77}$$

for all $\mathbf{e}(t_0)$. If the system under consideration is time invariant, the eigenvalues of the matrix $\mathbf{A} - \mathbf{LC}$ govern the transient behavior of the estimation error. These eigenvalues, referred to as observer poles, can be arbitrarily located in the complex plane by appropriate choice of the constant observer gain matrix $\mathbf{L}$ if and only if the system given by Eqs. (4) and (5) in Chapter 17 is completely reconstructible.[28] For LTI systems, reconstructibility is exactly equivalent to observability. Also, if the $\mathbf{L}$ matrix is to be real, the complex eigenvalues of $\mathbf{A} - \mathbf{LC}$ should be specified as conjugate pairs.

The discrete-time version of the observer is given below.[3] The structure of the observer to estimate the state $\mathbf{x}(k)$ of a linear, time-varying system described by Eqs. (10) and (11) in Chapter 17 is given by (Fig. 11)

$$\hat{\mathbf{x}}(k+1) = \mathbf{F}(k)\,\hat{\mathbf{x}}(k) + \mathbf{G}(k)\mathbf{u}(k) + \mathbf{L}(k)[\mathbf{y}(k) - \mathbf{C}(k)\,\hat{\mathbf{x}}(k) - \mathbf{D}(k)\mathbf{u}(k)] \tag{78}$$

The resulting equation for the state estimation error $\mathbf{e}(k)$ is

$$\mathbf{e}(k+1) \stackrel{\Delta}{=} \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) = [\mathbf{F}(k) - \mathbf{L}(k)\mathbf{C}(k)]\mathbf{e}(k) \tag{79}$$

with the initial condition

$$\mathbf{e}(k_0) = \mathbf{x}(k_0) - \hat{\mathbf{x}}(k_0) \tag{80}$$

If the observer is asymptotically stable

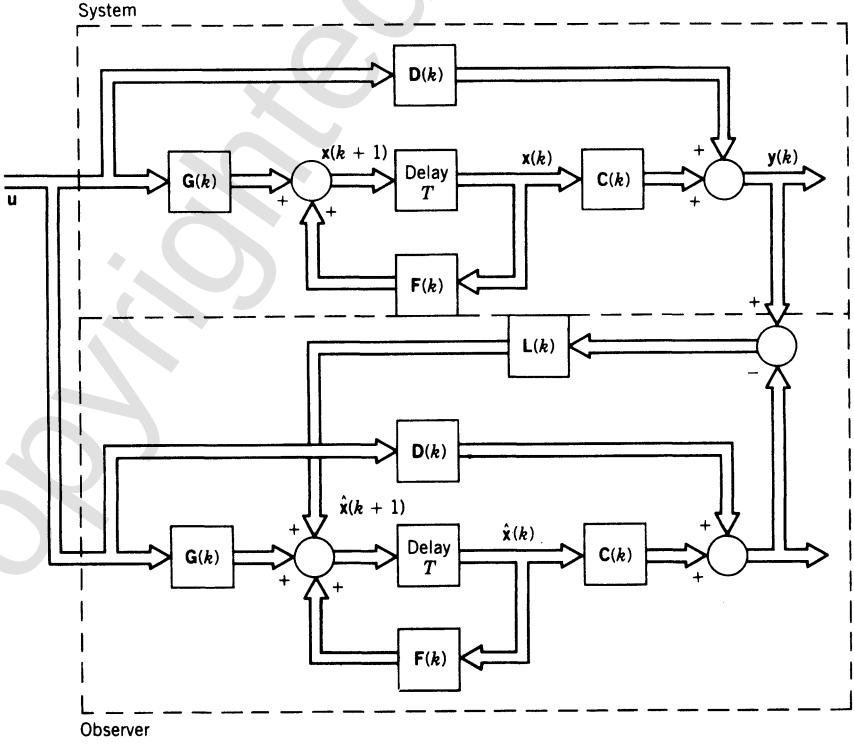


Figure 11 Observer for LTV discrete-time system.

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$$\lim_{k\to\infty} \mathbf{e}(k) = 0 \tag{81}$$

for all $\mathbf{e}(k_0)$. If the system under consideration is time invariant also, the eigenvalues of the matrix $\mathbf{F} - \mathbf{LC}$ govern the estimation error transients. Observability of the system given by Eqs. (12) and (13) in Chapter 17 is equivalent to arbitrary pole assignability for the observer.

For single-output systems, specification of the desired observer pole locations uniquely specifies the gain vector $\mathbf{L}$. Formulas for the gain vector $\mathbf{L}$, applicable to both continuous-time and discrete-time systems, will be given here. They are obtained by analogy with Eqs. (2)–(5). The object is to get the observer gain vector $\mathbf{L}$ so that the characteristic equation governing the estimation error transients has specified coefficients:

$$\begin{aligned}\beta_e(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) \\ &= \det(s\mathbf{I} - \mathbf{A}^{\mathrm{T}} + \mathbf{C}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}) \\ &= s^n + \sum_{i=0}^{n-1} \beta_{ie} s^i = 0\end{aligned} \tag{82}$$

where the matrix $\mathbf{A} - \mathbf{LC}$ is transposed so that the observer gain selection problem can be reduced to a form similar to the state feedback controller gain selection problem. Comparison of Eqs. (82) and (5) suggests that the following substitutions are necessary to convert algorithms for state feedback controller gain selection to observer gain selection.

CONTROLLER GAIN SELECTION		OBSERVER GAIN SELECTION
$\mathbf{A}$	$\longrightarrow$	$\mathbf{A}^{\mathrm{T}}$
$\mathbf{B}$	$\longrightarrow$	$\mathbf{C}^{\mathrm{T}}$
$\mathbf{K}$	$\longrightarrow$	$\mathbf{L}^{\mathrm{T}}$

(83)

Similarly, for discrete-time systems, we get the following transformations:

CONTROLLER GAIN SELECTION		OBSERVER GAIN SELECTION
$\mathbf{F}$	$\longrightarrow$	$\mathbf{F}^{\mathrm{T}}$
$\mathbf{G}$	$\longrightarrow$	$\mathbf{C}^{\mathrm{T}}$
$\mathbf{K}$	$\longrightarrow$	$\mathbf{L}^{\mathrm{T}}$

(84)

Using the preceding transformations, we get

$$\mathbf{L} = \beta_e(\mathbf{A}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{85}$$

for continuous-time systems and

$$\mathbf{L} = \beta_e(\mathbf{F}) \begin{bmatrix} \mathbf{C} \\ \mathbf{CF} \\ \vdots \\ \mathbf{CF}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{86}$$

for discrete-time systems. In the preceding equations,

$$\beta_e(\mathbf{A}) = \mathbf{A}^n + \sum_{i=0}^{n-1} \beta_{ie} \mathbf{A}^i \tag{87}$$

for continuous-time systems and $\beta_e(\mathbf{F})$ is a similar function of $\mathbf{F}$ for discrete-time systems. The β_{ie}'s are the coefficients of the desired characteristic equation (82) for the observer. A

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similar equation describes the discrete-time observer characteristic equation. If CACSD packages[4–6] supporting only controller pole placement algorithms are available, the transformations of Eqs. (83) and (84) are needed to select observer gains using these algorithms.

The observers of Eqs. (74) and (78) are called full-order observers since their dimensions are the same as those of the systems whose states are being estimated. Reduction of the observer dimension can be achieved by using the fact that the output equation provides us with p linear equations in the unknown state, where p is the number of output variables. Therefore, the observer need only provide $n - p$ additional linear equations and thus need only be of dimension $n - p$. For time-invariant systems, the corresponding observer gain matrix can be chosen to place the $n - p$ observer poles at any desired locations in the complex plane if the original systems of Eqs. (6) and (7) or (12) and (13) in Chapter 17 are completely observable. Equations for reduced-order observers for continuous-time systems are given by Kwakernaak and Sivan[3] and for discrete-time systems by Franklin and Powell.[11] In the presence of measurement noise, the state estimates $\mathbf{x}(t)$ or $\mathbf{x}(k)$ obtained using reduced-order observers are more sensitive than those obtained using full-order observers.

The discrete-time observer of Eq. (78) is also referred to as a prediction estimator since $\mathbf{x}(k + 1)$ is ahead of the last measurements used, $\mathbf{y}(k)$ and $\mathbf{u}(k)$. A variation of this observer, called the current estimator, is useful if the computation time associated with the observer is very short compared to the sampling interval for the sampled-data system. The corresponding observer equation for an LTI system is[11]

$$\hat{\mathbf{x}}(k + 1) = \mathbf{F}\,\hat{\mathbf{x}}(k) + \mathbf{Gu}(k + \mathbf{L}[y(k + 1) - \mathbf{CF}\hat{\mathbf{x}}(k) - \mathbf{CBu}(k) - \mathbf{Du}(k + 1)] \quad (88)$$

The state estimate $\hat{\mathbf{x}}(k + 1)$ therefore depends on the current measurements $\mathbf{y}(k + 1)$ and $\mathbf{u}(k + 1)$. In practice, however, the measurements precede the estimate by a very small computation time. The observer gain formula for a single-output system is then given by

$$\mathbf{L} = \beta_e(\mathbf{F}) \begin{bmatrix} \mathbf{CF} \\ \mathbf{CF}^2 \\ \vdots \\ \mathbf{CF}^n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (89)$$

instead of Eq. (86). The term $\beta_e(\mathbf{F})$ has the same meaning as before.

For multioutput LTI systems, specification of the desired observer poles does not specify the gain matrix $\mathbf{L}$ uniquely. The additional freedom in the gain matrix selection can be used to assign the eigenvectors (or generalized eigenvectors) of the matrix $\mathbf{A} - \mathbf{LC}$ or $\mathbf{F} - \mathbf{LC}$ in addition to the eigenvalues. The corresponding procedure would be very similar to that used for gain matrix selection for multi-input systems and described in Section 2. The transformations of Eqs. (83) and (84) can be used to adapt the controller gain selection procedure to observer gain selection. Alternatively, the observer gains can be chosen to design observers whose state estimates have low sensitivity to unmeasured disturbance inputs.[29]

The observer designs described work well in the absence of significant levels of measurement noise or unknown disturbance signals. The sensitivity of the state estimates to measurement noise and unknown disturbance signals depends on the specified location of the observer poles. If these pole locations are too far into the left half of the complex s-plane or too close to the origin in the complex z-plane, the state estimates would be unduly sensitive to measurement noise and disturbance signals. In the limiting case, the observers can be shown to reduce to ideal differentiators or differencing devices. The observer pole locations should therefore be chosen to avoid such high sensitivities of the state estimates but at the same time ensure that the estimation error transients decay more rapidly than the state variables being estimated. Specification of the observer poles in practice involves considerable trial and error in much the same manner as specification of closed-loop poles does for state feedback controller design. If some information is available concerning the distur-

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bance signals affecting the system and the measurement noise, the observer gain matrix selection problem can be formulated as an optimal-estimation problem.

5.2 The Optimal Observer

Optimization of observer design has been primarily performed assuming stochastic models for the disturbance inputs and measurement noise, though the effect of deterministic model errors on observer design is also important.[30] Consider the continuous-time system[3]

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{S}(t)\mathbf{w}_1(t) \qquad t \geq t_0 \tag{90}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{w}_2(t) \qquad t \geq t_0 \tag{91}$$

where $\mathbf{w}_1(t)$ is the random disturbance input, $\mathbf{w}_2(t)$ is the random measurement noise, and their joint probabilities are assumed to be known. The column vector $[\mathbf{w}_1^{\mathrm{T}}(t)\ \mathbf{w}_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is assumed to be a white-noise process with intensity

$$\mathbf{V}(t) = \begin{bmatrix} \mathbf{V}_1(t) & \mathbf{V}_{12}(t) \\ \mathbf{V}_{12}^{\mathrm{T}}(t) & \mathbf{V}_2(t) \end{bmatrix} \tag{92}$$

that is, the expected value

$$E\left\{ \begin{bmatrix} \mathbf{w}_1(t) \\ \mathbf{w}_2(t_1) \end{bmatrix} [\mathbf{w}_1^{\mathrm{T}}(t_2) \quad \mathbf{w}_2^{\mathrm{T}}(t_2)] \right\} = \mathbf{V}(t_1)\delta(t_1 - t_2) \tag{93}$$

where $\delta(t_1 - t_2)$ is the Dirac delta function.

The initial state $\mathbf{x}(t_0)$ is assumed to be a random variable uncorrelated with $\mathbf{w}_1$ and $\mathbf{w}_2$ and its probability given by

$$E[\mathbf{x}(t_0)] = \mathbf{x}_0 \quad \text{and} \quad E\{[\mathbf{x}(t_0) - \mathbf{x}_0][\mathbf{x}(t_0 - \mathbf{x}_0]^{\mathrm{T}}\} = \mathbf{Q}_0 \tag{94}$$

The observer form is given by Eq. (74) and Fig. 10. The optimal-observer problem consists of determining $\mathbf{L}(\tau)$, $t_0 \leq \tau \leq t$, and the initial condition on the observer $\hat{\mathbf{x}}(t_0)$ so as to minimize the expected value $E\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^{\mathrm{T}}\mathbf{W}(t)[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]\}$, where $\mathbf{W}(t)$ is a symmetric positive-definite weighting matrix.

If the problem as stated is nonsingular

$$\det[\mathbf{V}_2(t)] > 0 \qquad t \geq t_0 \tag{95}$$

and if the disturbance and measurement noise are uncorrelated

$$\mathbf{V}_{12}(t) = \mathbf{0} \tag{96}$$

the optimal-observer gain matrix is given by Kalman and Bucy[31] as

$$\mathbf{L}(t) = \mathbf{Q}(t)\mathbf{C}^{\mathrm{T}}(t)\mathbf{V}_2^{-1}(t) \qquad t \geq t_0 \tag{97}$$

where $\mathbf{Q}$ is a solution of the matrix Riccati equation:

$$\dot{\mathbf{Q}}(t) = \mathbf{A}(t)\mathbf{Q}(t) + \mathbf{Q}(t)\mathbf{A}^{\mathrm{T}}(t) + \mathbf{S}(t)\mathbf{V}_1(t)\mathbf{S}^{\mathrm{T}}(t) - \mathbf{Q}(t)\mathbf{C}^{\mathrm{T}}(t)\mathbf{V}_2^{-1}(t)\mathbf{C}(t)\mathbf{Q}(t) \qquad t \geq t_0 \tag{98}$$

with the initial condition

$$\mathbf{Q}(t_0) = \mathbf{Q}_0 \tag{99}$$

and the observer initial condition

$$\hat{\mathbf{x}}(t_0) = \mathbf{x}_0 \tag{100}$$

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The resulting state estimator is called the Kalman–Bucy filter. The similarity of Eqs. (97) and (98) to the corresponding Eqs. (25) and (26), respectively, for the LQR problem is a result of the duality of the state estimation and state feedback control problems noted earlier. One difference, however, is that the Riccati equation for the optimal observer can be implemented in real time since Eq. (99) is an initial condition for Eq. (98). In contrast, for the finite-time LQR problem, Eq. (27) gives the terminal condition for the Riccati equation (26).

The steady-state properties of the optimal observer for linear time-varying and time-invariant systems parallel those of the optimal controller for the LQR problem and are described by Kwakernaak and Sivan.[3] If the time-varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{S}(t)\mathbf{w}_1(t) \tag{101}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{w}_2(t) \tag{102}$$

is uniformly completely controllable by $\mathbf{w}_1(t)$ and uniformly completely reconstructible, the solution $\mathbf{Q}(t)$ of Eq. (98) converges to a steady-state solution $\mathbf{Q}_s(t)$ as $t_0 \to -\infty$ for any positive-semidefinite $\mathbf{Q}_0$. The corresponding steady-state optimal observer

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{L}_s(t)[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)] \tag{103}$$

where

$$\mathbf{L}_s(t) = \mathbf{Q}_s(t)\mathbf{C}^{\mathrm{T}}(t)\mathbf{V}_2^{-1}(t) \tag{104}$$

is exponentially stable. Also, if the system and the noise statistics are invariant, the matrix Riccati differential equation (98) becomes an algebraic equation as $t_0 \to -\infty$:

$$\mathbf{A}\mathbf{Q}_s + \mathbf{Q}_s\mathbf{A}^{\mathrm{T}} + \mathbf{S}\mathbf{V}_1\mathbf{S}^{\mathrm{T}} - \mathbf{Q}_s\mathbf{C}^{\mathrm{T}}\mathbf{V}_2^{-1}\mathbf{C}\mathbf{Q}_s = \mathbf{0} \tag{105}$$

If the corresponding time-invariant system is completely controllable by the input $\mathbf{w}_1(t)$ and completely observable, Eq. (105) has a unique positive-definite solution $\mathbf{Q}_s$ and the corresponding steady-state optimal observer of Eqs. (103) and (104) is asymptotically stable. Note that the measurable input $\mathbf{u}(t)$ has been omitted from Eqs. (101) and (102) for simplicity but does not change the substance of the results.

The discrete-time version of the optimal linear observer follows.[3] Consider the discrete-time system

$$\mathbf{x}(k+1) = \mathbf{F}(k)\mathbf{x}(k) + \mathbf{G}(k)\mathbf{u}(k) + \mathbf{S}(k)\mathbf{w}_1(k) \tag{106}$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) + \mathbf{w}_2(k) \tag{107}$$

where $\mathbf{w}_1(k)$, $\mathbf{w}_2(k)$ are zero-mean, uncorrelated vector random variables representing disturbance and measurement noise, respectively. Their joint probabilities are assumed to be known. The column vector $[\mathbf{w}_1^{\mathrm{T}}(k)\ \mathbf{w}_2^{\mathrm{T}}(k)]^{\mathrm{T}}$ has the variance matrix

$$E\left\{\begin{bmatrix}\mathbf{w}_1(k)\\ \mathbf{w}_2(k)\end{bmatrix}[\mathbf{w}_1^{\mathrm{T}}(k) \quad \mathbf{w}_2^{\mathrm{T}}(k)]\right\} = \begin{bmatrix}\mathbf{V}_1(k) & \mathbf{V}_{12}(k)\\ \mathbf{V}_{12}^{\mathrm{T}}(k) & \mathbf{V}_2(k)\end{bmatrix} \tag{108}$$

The initial state $\mathbf{x}(k_0)$ is considered to be a random variable, uncorrelated with $\mathbf{w}_1$ and $\mathbf{w}_2$ with

$$E[\mathbf{x}(k_0)] = \mathbf{x}_0 \quad \text{and} \quad E\{(\mathbf{x}(k_0) - \mathbf{x}_0)(\mathbf{x}(k_0) - \mathbf{x}_0)^{\mathrm{T}}\} = \mathbf{Q}_0 \tag{109}$$

The observer form is given by Eq. (78) and Fig. 11. The optimal-observer problem consists of determining $\mathbf{L}(k_\tau)$, $k_0 \le k_\tau \le k$, and initial condition on the observer $\hat{\mathbf{x}}(k_0)$ so as to minimize the expected value $E\{[\mathbf{x}(k) - \hat{\mathbf{x}}(k)]\mathbf{W}(k)[\mathbf{x}(k) - \hat{\mathbf{x}}(k)^{\mathrm{T}}]\}$, where $\mathbf{W}(k)$ is a symmetric positive-definite weighting matrix.

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If the problem as stated is nonsingular

$$\det[\mathbf{V}_2(k)] > 0 \qquad k \geq k_0 \tag{110}$$

the optimal-observer gain matrix is given by the recurrence relations

$$\mathbf{L}(k) = [\mathbf{F}(k)\mathbf{Q}(k)\mathbf{C}^{\mathrm{T}}(k) + \mathbf{V}_{12}(k)][\mathbf{V}_2(k) + \mathbf{C}(k)\mathbf{Q}(k)\mathbf{C}^{\mathrm{T}}(k)] \tag{111}$$

$$\mathbf{Q}(k+1) = [\mathbf{F}(k) - \mathbf{L}(k)\mathbf{C}(k)]\mathbf{Q}(k)\mathbf{F}^{\mathrm{T}}(k) + \mathbf{V}_1(k) - \mathbf{L}(k)\mathbf{V}_{12}^{\mathrm{T}}(k) \tag{112}$$

with the initial condition

$$\mathbf{Q}(k_0) = \mathbf{Q}_0 \tag{113}$$

and $k \geq k_0$. The initial condition on the observer state should be

$$\hat{\mathbf{x}}(k_0) = \mathbf{x}_0 \tag{114}$$

Again, the similarity of the optimal-observer equations (111)–(113) to the optimal-controller equations (33)–(35) for the LQR problem results from the duality of state estimation and state feedback control problems.

The similarity extends to the steady-state behavior of the optimal observer and the optimal controller.[3] If the time-varying system

$$\mathbf{x}(k+1) = \mathbf{F}(k)\mathbf{x}(k) + \mathbf{S}(k)\mathbf{w}_1(k) \tag{115}$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{w}_2(k) \tag{116}$$

is uniformly completely controllable by $\mathbf{w}_1(k)$ and uniformly completely reconstructible, the solution $\mathbf{Q}(k)$ of Eqs. (111) and (112) converges to a steady-state solution $\mathbf{Q}_s(k)$ as $k_0 \rightarrow -\infty$ for any positive-semidefinite $\mathbf{Q}_0$. The corresponding steady-state observer

$$\hat{\mathbf{x}}(k+1) = \mathbf{F}(k)\,\hat{\mathbf{x}}(k) + \mathbf{L}_s(k)[\mathbf{y}(k) - \mathbf{C}(k)\,\hat{\mathbf{x}}(k)] \tag{117}$$

where $\mathbf{L}_s(k)$ is obtained using $\mathbf{Q}_s(k)$ for $\mathbf{Q}(k)$ in Eq. (111) is exponentially stable. Also, if the system and the noise statistics are time invariant, the matrix difference equations (111) and (112) become algebraic equations as $k_0 \rightarrow -\infty$. If the corresponding time-invariant system is completely controllable by the input $\mathbf{w}_1(k)$ and completely observable, the resulting steady-state optimal observer is asymptotically stable. Again, note that the measurable input $\mathbf{u}(k)$ has been omitted from Eqs. (115) and (116) for simplicity but does not change the substance of the results.

Interested readers are referred to Kwakernaak and Sivan[32] for a more complete consideration of the optimal observer. There is also extensive literature available on Kalman filters.[31,33,34]

6 OBSERVER-BASED CONTROLLERS

The observers described in the preceding section can be used to provide estimates of system state that, in turn, can be used to provide state feedback as described in Sections 2–4. The resulting controllers are dynamic compensators and are referred to as observer-based controllers.

The design of such observer-based controllers for LTI systems is simplified somewhat by the fact that their modes or eigenvalues satisfy the separation property, that is, the eigenvalues of the observer-controller are the same as the eigenvalues of the observer and the eigenvalues of the controller, the latter evaluated assuming perfect state measurement. For

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continuous-time LTI systems described by Eqs. (6) and (7) in Chapter 17, the control law given by Eq. (1), and the observer given by Eq. (74) with constant coefficient matrices, the observer-controller is given by Fig. 12 and the characteristic equation of the corresponding closed-loop system is[7]

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) = 0 \tag{118}$$

A similar result can be shown to be true for discrete-time LTI systems.[11] The corresponding closed-loop system is shown in Fig. 13 and has the characteristic equation

$$\det(z\mathbf{I} - \mathbf{F} + \mathbf{GK})\det(z\mathbf{I} - \mathbf{F} + \mathbf{LC}) = 0 \tag{119}$$

For LTI systems subjected to unmeasured randomly varying disturbance inputs and measurement errors, if the statistics of these signals are known, state estimators of the Kalman–Bucy type will be used. The resulting estimator-based controllers have eigenvalues that also satisfy a separation property. As a result of the separation property for controllers based on observers or Kalman–Bucy filters, the design of the controllers can be treated independently of the observer.

The use of observers or Kalman filters to provide state estimates for state feedback controllers does, however, impair overall controller performance. For instance, the transient

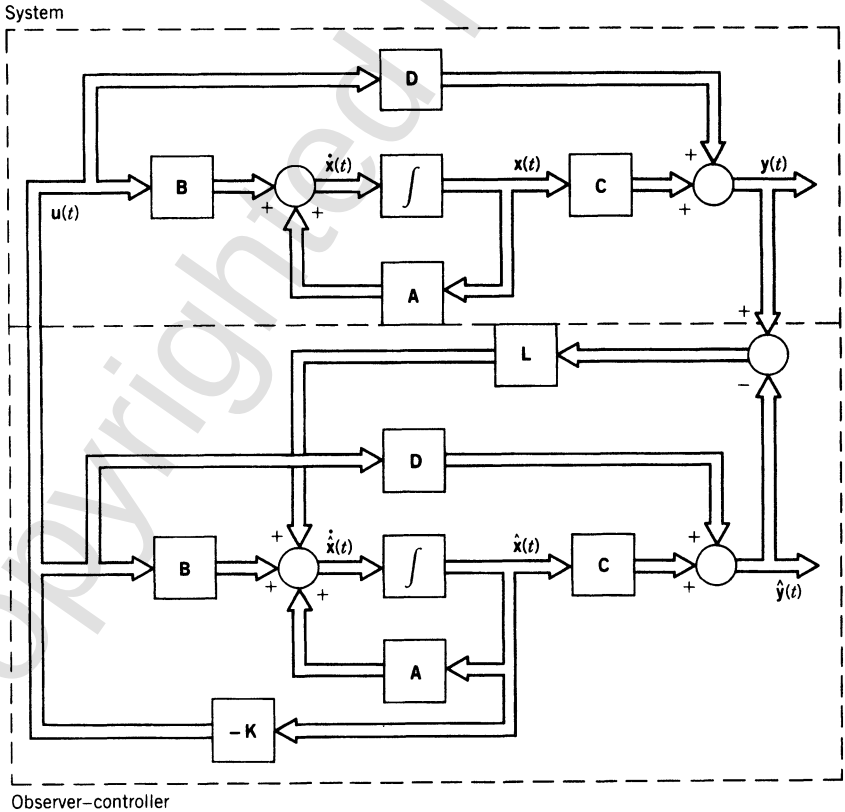


Figure 12 Observer-based controller for LTI continuous-time system.

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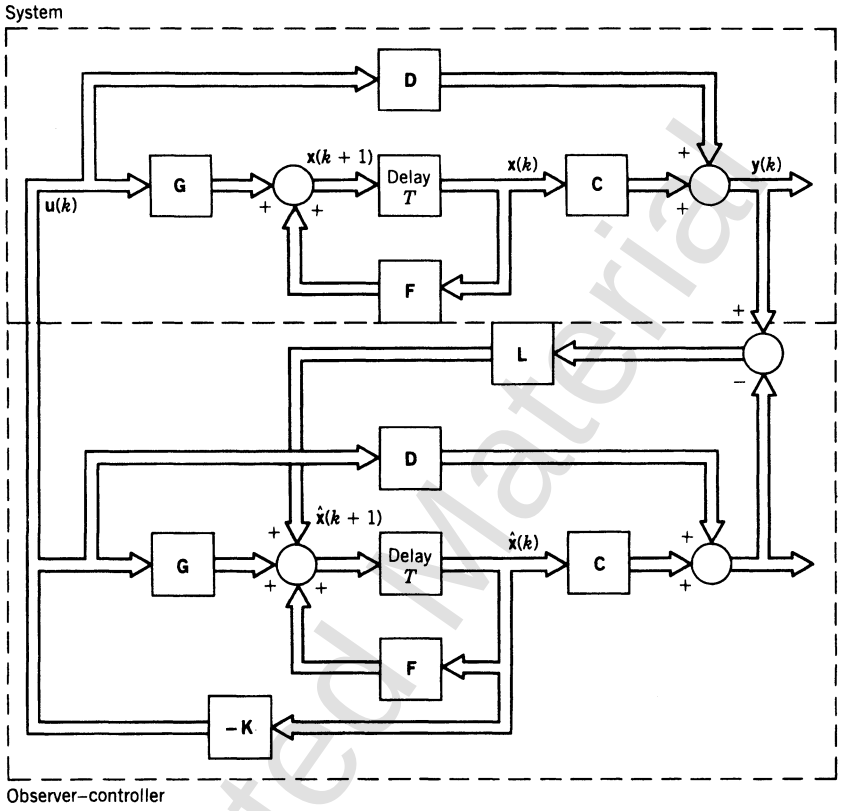


Figure 13 Observer-based controller for LTI discrete-time system.

response of such controllers is poorer than that of controllers using complete state feedback. Moreover, the properties relating to the gain and phase margins of optimal controllers for the LQR problem are not applicable if the controllers use estimated states for feedback rather than measured states.[13] There is a definite loss of controller robustness associated with the use of state estimators. The robustness of such controllers is more properly evaluated by considering them as dynamic compensators and using methods common to the frequency-domain approach.[35]

Observer-based controllers for LTI systems can naturally be examined using transfer function or frequency-domain-based methods. Such a linking of time-domain- and frequency-domain-based controller designs offers a number of advantages. The state-space-based design approach leads to consideration of controller structures that are not obvious from a transfer function approach. In addition, the state-space approach alerts the designer to the problem of loss of controllability or observability via pole–zero cancellation. On the other hand, the transfer function approach can result in controllers that cannot be obtained by using observer-based controllers.[7] In addition, the consideration of observer-based controllers from a transfer function perspective helps evaluate the controllers to see whether proven and practical design guidelines are violated. Such guidelines invariably use transfer function terminology since they have evolved from years of experience using classical control techniques. If such guide-

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lines are violated, the state-space-based controller design procedure can be modified appropriately.[36]

A recent example of a MIMO controller design method that has evolved from a combination of frequency-domain methods and state-space methods is the linear-quadratic-Gaussian method with loop-transfer-recovery (LQG/LTR), developed by Athans.[37] The procedure relies on the fact that results and requirements relating to control system robustness to modeling errors are best presented in the frequency domain. The powerful controller and estimator structures resulting from LQR formulations of the control and state estimation problem are used. These structures are useful in this design method because their robustness and performance have been well studied, using frequency-domain measures.[14] The resulting method relies upon designer expertise in formulating good performance specifications at the outset. The design method therefore avoids the main weakness of state-space methods—namely, the weak connection between performance measures used by these methods and performance measures of engineering significance. The computation of the controller is, however, straightforward in this method, since the controller structures are derived from well-established state-space methods and are well supported by commercial CACSD packages.[4–6]

The first step in the design method is the definition of the design plant model. This model includes not only the nominal model of the system to be controlled but also scaling of the variables and augmentation of the dynamics, such as the inclusion of integrators dictated by control objectives. The number of control inputs r and the number of outputs p are assumed to be equal in the following development. The model is also linear and time invariant and strictly proper, that is, the transmission matrix $\mathbf{D}$ in the system equations (6) and (7) in Chapter 17 is zero. The transfer function matrix of the system is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} \tag{120}$$

Modeling inaccuracy is treated as follows. The actual transfer function matrix is given by

$$\mathbf{H}_A(s) = [\mathbf{I}_n + \mathbf{E}(s)]\mathbf{H}(s) \tag{121}$$

where $\mathbf{E}(s)$ characterizes the modeling error. The maximum singular value $\sigma_{\max}$ of the matrix $\mathbf{E}(j\omega)$ is assumed to be bounded by a known bound $e_m(\omega)$:

$$\sigma_{\max}[\mathbf{E}(i\omega)] < e_m(\omega) \tag{122}$$

The second step in the design procedure is the specification of a target feedback loop that has satisfactory robustness, stability, and performance specifications. The target feedback loop is shown in Fig. 14. It is obviously not directly implementable since the control inputs $\mathbf{u}$ do not appear in the system. The matrix $\mathbf{L}$ is a constant matrix and is chosen as described later. It is the designer's task to experiment with different choices of $\mathbf{L}$ and evaluate whether

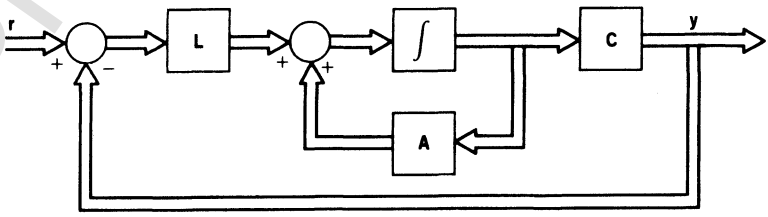


Figure 14 Target feedback loop block diagram.

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the resulting system has satisfactory performance. This stage of the design therefore requires considerable trial and error.

Athans[36] has suggested using the steady-state Kalman–Bucy filter formulation with stationary system and noise characteristics. It should be noted, however, that the formulation is not used here to perform an estimation task. It is being used here to help calculate the matrix **L** because the resulting target feedback loop has well-known performance and robustness characteristics. At the minimum, selection of **L** as described by Athans[36] guarantees that the target feedback loop will not amplify disturbances entering the system at the output. In addition, the target feedback 1oop will not go unstable as long as the modeling uncertainty $e_m(\omega)$ in Eq. (122) is below 0.5.

Once the target feedback loop is chosen, the final step in the design procedure is the design of a compensator to enable the controlled system to approximate the behavior of the target feedback loop closely. Athans[36] has proposed the compensator structure shown in Fig. 15. The similarity of the controller to the observer-based controller in Fig. 12 is clear. The gain matrix **K** is computed via the solution of a version of the LQR problem. The loop transfer recovery (LTR) result, credited by Athans to other researchers, guarantees that for minimum-phase systems the procedure described yields a controlled system behavior that approximates the target feedback loop behavior as closely as desired.

For non-minimum-phase systems, the design procedure remains the same. The only difference is that the final design may not approximate the behavior of the target feedback loop closely. In effect, this would result in additional design iterations to arrive at a satisfactory final design.

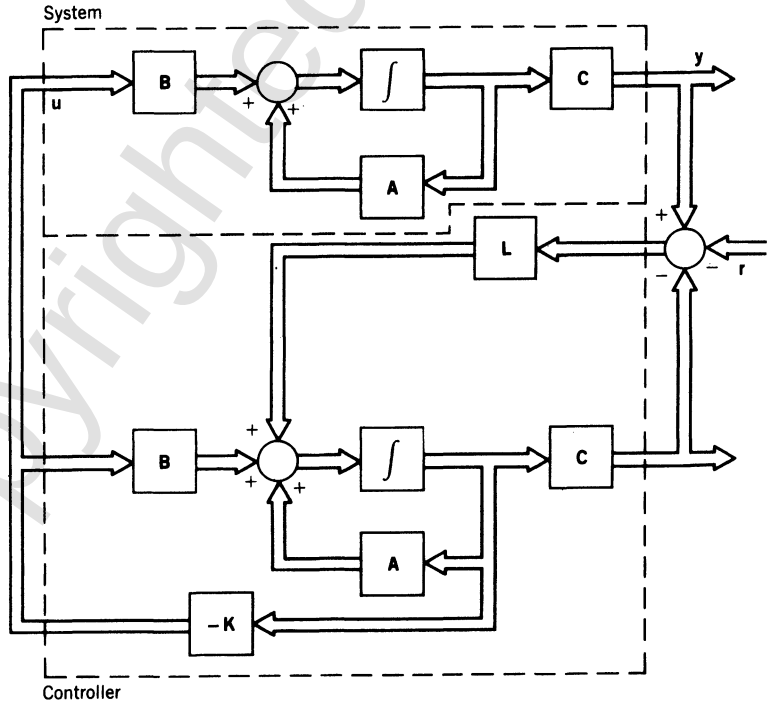


Figure 15 Compensator structure for LQR/LTR method.

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The LQG/LTR design procedure has been applied successfully to evaluate the feasibility of MIMO control for aircraft and helicopter flight control, jet engine control, and submersible control.

7 CONCLUSION

The primary emphasis in this chapter on linear time-invariant finite-dimensional systems is a reflection of the state of the literature on the subject and the practice of the art. The reader is referred to the following works for more exhaustive treatment of some of the topics not covered at great length here. The subject of multivariable control using state-space methods has been addressed at much greater length by Kailath[7] and others.[38] The application of state-space methods to nonlinear system analysis and control is treated at some length by Ramnath et al.[32] Optimal-control problems other than the LQR formulation have been described in detail in a number of textbooks.[39,40] The subject of adaptive control refers to control situations where the controller parameters are adapted or adjusted as the behavior of the system being controlled changes. One approach to adaptive control, termed the Model Reference Approach and employing state-space description, has been described at length by Landau.[41] Distributed-parameter systems are examples of systems with infinite-dimensional states. Application of state-space methods to these systems has been described by Tzafestas et al.[42] Time-delayed systems are also examples of systems with infinite-dimensional states. The analysis and control of such systems and of many of the other types of systems referred to in this section remains a subject of current research. For current research results in these areas, the reader is referred to journals such as the *ASME Journal of Dynamic Systems, Measurement and Control, IEEE Transactions on Automatic Control, AIAA Journal of Guidance, Control and Dynamics, SIAM Journal on Control, and Automatica, The Journal of the International Federation of Automatic Control.*

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